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Stability and asymptotic analysis of the Föllmer–Schweizer decomposition on a finite probability space

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First, we consider the problem of hedging in complete binomial models. Using the discrete-time Föllmer–Schweizer decomposition, we demonstrate the equivalence of the backward induction and sequential regression approaches. Second, in incomplete trinomial models, we examine the extension of the sequential regression approach for approximation of contingent claims. Then, on a finite probability space, we investigate stability of the discrete-time Föllmer–Schweizer decomposition with respect to perturbations of the stock price dynamics and, finally, perform its asymptotic analysis under simultaneous perturbations of the drift and volatility of the underlying discounted stock price process, where we prove stability and obtain explicit formulas for the leading-order correction terms.

1. Introduction

In practice, financial models are not exact — as in any field, modeling based on real data introduces some degree of error. Therefore, it is important to understand the effect error has on the calculations and assumptions we make on the model. In this paper, we focus on the stability and asymptotic analysis of the Föllmer–Schweizer decomposition, as among the pricing and hedging approaches, in incomplete markets, it gives the best approximation of a given contingent claim in the sense of the least-squares error, which is one of the most natural criteria used in practice. Further, in complete models, the Föllmer–Schweizer decomposition is consistent with the backward induction, which is another canonical method in financial mathematics.

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Steven E. Shreve [2004a] introduces option pricing in a highly accessible manner. His text predominantly focuses on the binomial model, and in this paper, we go beyond it, as there are many models used in practice that are not binomial. As the most natural extension of a binomial model is a trinomial one, below, we also give it special consideration. Note that both binomial and trinomial models, despite their simplicity, are widely used in approximations of pricing and hedging in more advanced models, including the continuous-time ones; see, e.g., [Brigo and Mercurio 2006].

In this paper, we extend the introduction to asset hedging given by Shreve to the strategy of sequential regression, keeping the discrete-time framework but allowing for consideration of other market models, including incomplete ones. In the complete case, we show that the strategy of sequential regression introduced by Föllmer and Schweizer [1989] is equivalent to Shreve's recursive hedging formula. We then extend our discussion to the incomplete trinomial model, after which we show small market perturbations have a little effect, which we quantify, on hedging strategies and option pricing, and derive formulas for correction factors.

The remainder of this paper is organized as follows. In Section 2, we formulate the minimization problem and define the Föllmer–Schweizer decomposition. Section 3 contains its investigation in complete binomial markets, where we also prove the equivalence of the approach based on the Föllmer–Schweizer decomposition to the backward induction. Section 4 presents the discussion of the general incomplete case. In Section 5, we revisit the stability question in the context of perturbations of the model parameters, where we introduce a parametrization of perturbations that allows for simultaneous distortions of its drift and volatility of the underlying stock price process. We prove the stability of the Föllmer–Schweizer decomposition under such perturbations. Finally, in Section 6, we obtain explicit formulas for the first-order correction terms of each component of the decomposition under such perturbations, including the correction to the optimal trading strategy.

2. The discrete-time Föllmer–Schweizer decomposition

Let $(\Omega, \mathbb{P})$ be a finite probability space, N a fixed positive integer and $\mathbb{F} = (\mathcal{F}_n)_{n=0,1,\dots,N}$ a filtration, i.e., an increasing family of subalgebras, each containing Ω and $\varnothing$. Assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_N$ contains all subsets of Ω. As we work on a finite probability space, without loss of generality, we suppose that $\mathbb{P}[\omega] > 0$ for every $\omega \in \Omega$. We suppose that there is a bank account, which we will use as a numéraire, and in particular, that its price process is equal to 1 at all times. Let $S = (S_n)_{n=0,1,\dots,N}$ be a real-valued, $\mathbb{F}$-adapted process; i.e., each S_n is $\mathcal{F}_n$-measurable. S describes the discounted price process of a stock. We denote by

$$\Delta S_n := S_n - S_{n-1} \quad \text{for } n = 1, \dots, N$$

the increments of S. We call a process $\vartheta = (\vartheta_n)_{n=1,\dots,N}$ *predictable* if ϑ_n is $\mathcal{F}_{n-1}$-measurable for each n. Let Θ be the set of all predictable processes ϑ that financially correspond to self-financing trading strategies, in view of the presence of money market account, which we use as a numéraire.

Definition 2.1. For $\vartheta \in \Theta$, we define the process

$$G_n(\vartheta) := \sum_{j=1}^{n} \vartheta_j \Delta S_j \quad \text{for } n = 0, 1, \dots, N.$$

For a given a random variable V_N and $c \in \mathbb{R}$, one can consider the following problem posed in [Schweizer 1995]:

$$\text{minimize } \mathbb{E}[(V_N - c - G_N(\vartheta))^2] \text{ over all } \vartheta \in \Theta \text{ and } c \in \mathbb{R}. \tag{1}$$

Interpretation 2.2. As we view S_n as the price at time n of a risky financial asset, the process ϑ describes the trading strategy of some investor in the market, where ϑ_n is the number of shares of stock held between the times n and $n+1$. The process $G(\vartheta)$ becomes the *gains from the trade* process. We now interpret V_N as a *nontraded security* measured in the units of the bank account with maturity N and c as the *initial capital*. Thus problem (1) can be interpreted as finding a self-financing trading strategy that gives the best least-squares approximation of V_N. Mathematically, (1) is also closely related to the problem considered in [Schweizer 1994], finding the best approximation of a random variable by a stochastic integral (plus a constant). Quadratic optimization problems of the form (1) also appear in the asymptotic analysis of stochastic control problems with respect to perturbations of the initial data, where they govern the second-order correction terms; see [Kramkov and Sîrbu 2006a; 2006b, Mostovyi and Sîrbu 2019; Mostovyi 2020] for details.

A solution (1), given in terms of an explicit formula for an optimal trading strategy $\hat{\vartheta}$, is known as *sequential regression* and is shown in [Föllmer and Schweizer 1989]. For a general probability space, such a solution is subject to additional conditions on S and is closely related to the discrete-time *Föllmer–Schweizer decomposition*, defined below.

Definition 2.3. We use the definition of the *nondegeneracy condition* (ND) as given in [Schweizer 1995]; that is, S satisfies (ND) if there exists a constant $\delta \in (0, 1)$ such that

$$(\mathbb{E}[\Delta S_n \mid \mathcal{F}_{n-1}])^2 \le \delta \mathbb{E}[\Delta S_n^2 \mid \mathcal{F}_{n-1}] \quad \mathbb{P}\text{-a.s. for } n = 1, \dots, N.$$

Remark 2.4. We note that on finite probability spaces, (ND) holds in nontrivial (or rather nondegenerate) cases.

Definition 2.5. We now introduce the *discrete Föllmer–Schweizer decomposition*, following [Schweizer 1995]. Let

$$S = M + A$$

be the semimartingale decomposition of S into a martingale M and a predictable process A. The random variable V_N admits the *discrete Föllmer–Schweizer decomposition* if it can be written as

$$V_N = V_0 + \sum_{j=1}^{N} \hat{\vartheta}_j \Delta S_j + L_N \tag{2}$$

for some $V_0 \in \mathbb{R}$, a process $\hat{\vartheta} \in \Theta$, and a $\mathbb{P}$-martingale L, such that

(i) L and M are orthogonal, i.e., LM is a $\mathbb{P}$-martingale, and

(ii) $\mathbb{E}[L_0] = 0$.

Note that when $\mathcal{F}_0$ is trivial, the latter condition reads $L_0 = 0$.

Using the sequential regression approach, following [Föllmer and Schweizer 1989], we obtain the following formula for an optimal hedging strategy $\hat{\vartheta}$:

$$\hat{\vartheta}_n := \frac{\mathrm{Cov}_{\mathcal{F}_{n-1}}\big[V_N - \sum_{j=n+1}^{N} \hat{\vartheta}_j \Delta S_j,\, \Delta S_n\big]}{\mathrm{Var}_{\mathcal{F}_{n-1}}[\Delta S_n]}, \quad n = 1, \dots, N, \tag{3}$$

where $\mathrm{Cov}_{\mathcal{F}_{n-1}}[\,\cdot\,,\cdot\,]$ and $\mathrm{Var}_{\mathcal{F}_{n-1}}[\,\cdot\,]$ denote the conditional covariance and variance, respectively. This demonstrates the richness of the FS-decomposition as an analytic tool. With very limited assumptions about V_N, we are able to obtain an explicit formula for the optimal (in the sense of (1)) hedging strategy. Furthermore, this hedging formula holds in both complete and incomplete markets, which are discussed in the following sections.

3. Complete markets

In the settings of the previous section, when the market is complete, every contingent claim V_N can be represented as

$$V_N = V_0 + \sum_{j=1}^{N} \vartheta_j \Delta S_j$$

for some $\vartheta \in \Theta$ and $V_0 \in \mathbb{R}$. Note that this situation corresponds to $L = 0$ in Definition 2.5. Put differently, V_N can now be obtained by trading between the money market account and the stock.

Binomial asset pricing model. We now introduce a simple framework for the problem following [Shreve 2004a], however directly using the bank account as a

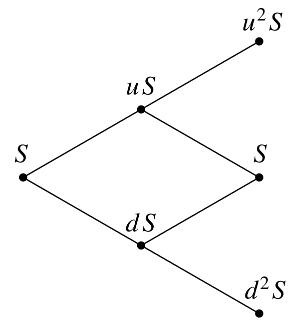

Figure 1. Example of a 2-period binomial model.

numéraire. Consider a *binomial asset pricing model*, where at each time step k, S_{k+1} can take one of two values, uS_k or dS_k, with probabilities $p \in (0, 1)$ and $q := 1 - p$, and where $u > 1$ and $d \in (0, 1)$, known as the *up factor* and *down factor*, respectively, are fixed positive constants with $u > d$. The value at each time k is determined by a (not necessarily fair) coin flip ω_k, which can take either the value H or T and is independent of other coin tosses. Let $\mathbb{F} = (\mathcal{F}_n)_{n=0,\dots,N}$ be the filtration, where each $\mathcal{F}_n$ contains information about the first n coin tosses. An example of a 2-period binomial model is shown in Figure 1.

Remark 3.1. For Figure 1, for the binomial model, we considered the case when $d = 1/u$. That is, after an even number of time steps, the stock price returns to its original value if we flip exactly the same number of heads and tails. However, in general, it is not necessarily the case that $u = 1/d$.

Let X be the replicating wealth process for the contingent claim V_N, i.e., a self-financing process starting from the initial wealth X_0 and such that

$$X_N = V_N.$$

Classical backward induction approach, for which we refer to [Shreve 2004a], assures that the number of shares of stock in the replicating portfolio for V_N held between times n and $n+1$, can be obtained via the formula

$$\vartheta_{n+1}(\omega) = \frac{X_{n+1}(\omega H) - X_{n+1}(\omega T)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)}, \tag{4}$$

where $\omega = \omega_1, \dots, \omega_n$. This result is known as the delta-hedging rule; see [Shreve 2004a, Theorem 1.2.2, p. 12].

The main result of this section is proving that in complete binomial settings the delta-hedging rule gives the same strategy as the Föllmer–Schweizer decomposition.

Proposition 3.2. *In complete binomial settings*, *formulas* (3) *and* (4) *are equivalent.*

Proof. Below $\hat{\vartheta}$ will denote the strategy obtained from formula (3), and ϑ will denote the replicating strategy given by (4) for each n. First, for $n = N$, (3) reads

$$\hat{\vartheta}_N = \frac{\mathrm{Cov}_{\mathcal{F}_{N-1}}[V_N, \Delta S_N]}{\mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S_N]} = \frac{\mathrm{Cov}_{\mathcal{F}_{N-1}}[X_N, \Delta S_N]}{\mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S_N]}, \tag{5}$$

as $X_N = V_N$. Also, since[1]

$$X_{n+1} = X_n + \vartheta_{n+1}(S_{n+1} - S_n), \quad n = 0, \dots, N-1,$$

we have

$$X_N - \mathbb{E}_{\mathcal{F}_{N-1}}[X_N] = \vartheta_N(\Delta S_N - \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N]).$$

Therefore, we can rewrite $\mathrm{Cov}_{\mathcal{F}_{N-1}}[V_N, \Delta S_N]$ as

$$\begin{aligned}
\mathrm{Cov}_{\mathcal{F}_{N-1}}[V_N, \Delta S_N] &= \mathbb{E}_{\mathcal{F}_{N-1}}[(X_N - \mathbb{E}_{\mathcal{F}_{N-1}}[X_N])(\Delta S_N - \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N])] \\
&= \vartheta_N \mathbb{E}_{\mathcal{F}_{N-1}}[(\Delta S_N - \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N])(\Delta S_N - \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N])] \\
&= \vartheta_N \,\mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S_N].
\end{aligned}$$

And thus

$$\vartheta_N = \frac{\mathrm{Cov}_{\mathcal{F}_{N-1}}[V_N, \Delta S_N]}{\mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S_N]}.$$

Comparing ϑ_N to $\hat{\vartheta}_N$ from (5), we deduce that they coincide. This also implies

$$X_{N-1} = X_N - \hat{\vartheta}_N \Delta S_N.$$

Or equivalently, we have

$$X_N - \sum_{j=N}^{N} \hat{\vartheta}_N \Delta S_N = X_{N-1}. \tag{6}$$

The latter expression, however, is exactly the term appearing in $\hat{\vartheta}_{N-1}$ in (3), which via (6), we can rewrite as

$$\hat{\vartheta}_{N-1} = \frac{\mathrm{Cov}_{\mathcal{F}_{N-2}}\big[X_N - \sum_{j=N}^{N} \hat{\vartheta}_N \Delta S_{N-1}, \Delta S_{N-1}\big]}{\mathrm{Var}_{\mathcal{F}_{N-2}}[\Delta S_{N-1}]} = \frac{\mathrm{Cov}_{\mathcal{F}_{N-2}}[X_{N-1}, \Delta S_{N-1}]}{\mathrm{Var}_{\mathcal{F}_{N-2}}[\Delta S_{N-1}]},$$

which differs from (5) only by the value of the index. Therefore, line by line applying the argument above used for proving that $\vartheta_N = \hat{\vartheta}_N$, we can show that $\vartheta_{N-1} = \hat{\vartheta}_{N-1}$. Proceeding in such a way, we can show that $\vartheta_n = \hat{\vartheta}_n$ for each $n \in \{0, \dots, N\}$. □

Remark 3.3. Notice that Proposition 3.2 only demonstrates the equivalence of the specific strategies of backward recursion and sequential regression in the binomial model, not necessarily the uniqueness of the minimizing strategy. However, the

[1]We use indices for ϑ in a way consistent with [Schweizer 1995], so that ϑ is predictable.

uniqueness of the solution to (1), that is, of the optimal stochastic integral, the associated strategy, and a constant follows from the strict convexity of the quadratic objective in (1) and some computations under the risk-neutral measure.

Example 3.4. Consider a 3-step binomial asset pricing model with $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$, $p = \frac{1}{2}$, $q = \frac{1}{2}$, and a *European call option* expiring at time $N = 3$ with strike price $K = 1$. Note that in this market, the one-step risk-neutral probabilities are $\tilde{p} = \tilde{q} = \frac{1}{2}$ and the nonzero interest rate can be handled by considering the discounted stock. We will illustrate the optimal hedge using both (3) and (4). Recall that the value at time N of a European call is given by

$$V_N = (S_N - K)^+. \tag{7}$$

We compute the stock prices and discounted asset values at each time step for each possible combination of coin flips using the formulas

$$\begin{aligned} S_n(\omega_1, \dots, \omega_n) &= S_0 * u^{(\#\text{ heads})} * d^{(\#\text{ tails})}, \\ V_n(\omega_1, \dots, \omega_n) &= \left(\frac{1}{1+r}\right)\tilde{\mathbb{E}}[V_{n+1} \mid \omega_1, \dots, \omega_n], \end{aligned}$$

where $\mathbb{P}$ is the risk-neutral probability measure. The trees of stock prices and discounted asset values are shown in Figure 2. For brevity, we will calculate a hedge using both formulas at time 2, since $A_N = V_N$ in the sequential regression formula at the penultimate time step. We aim to show that $\vartheta_2(TT) = \xi_2(TT)$. Calculating $\vartheta_2(TT)$ using (4) yields

$$\frac{1-0}{2-\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}.$$

Using (3), $\xi_2(TT)$ is given by

$$\begin{aligned} \frac{\mathrm{Cov}_{\mathcal{F}_2}[V_3, \Delta S_3 \mid TT]}{\mathrm{Var}_{\mathcal{F}_2}[(\Delta S_3 \mid TT)]} &= \frac{\mathbb{E}_{\mathcal{F}_2}[(V_3 - \mathbb{E}_{\mathcal{F}_2}[V_3 \mid TT])(\Delta S_3 - \mathbb{E}_{\mathcal{F}_2}[\Delta S_3 \mid TT]) \mid TT]}{\mathbb{E}_{\mathcal{F}_2}[(\Delta S_3 - \mathbb{E}_{\mathcal{F}_2}[\Delta S_3 \mid TT])^2 \mid TT]} \\ &= \frac{\mathbb{E}_{\mathcal{F}_2}\big[\big(V_3 - \big(\frac{1}{2}*1+\frac{1}{2}*0\big)\big)\big(\Delta S_3 - \big(\frac{1}{2}*1+\frac{1}{2}*-\frac{1}{2}\big)\big) \mid TT\big]}{\mathbb{E}_{\mathcal{F}_2}[\Delta S_3^2 \mid TT] - \mathbb{E}_{\mathcal{F}_2}[\Delta S_3 \mid TT]^2} \\ &= \frac{\mathbb{E}_{\mathcal{F}_2}\big[\big(V_3 - \frac{1}{2}\big)\big(\Delta S_3 - \frac{1}{4}\big) \mid TT\big]}{\frac{5}{8} - \frac{1}{16}} \\ &= \frac{\big(\frac{1}{2}\big)\big(1-\frac{1}{2}\big)\big(1-\frac{1}{4}\big) + \big(\frac{1}{2}\big)\big(0-\frac{1}{2}\big)\big(-\frac{1}{2}-\frac{1}{4}\big)}{\frac{9}{16}} \\ &= \frac{\frac{3}{16}+\frac{3}{16}}{\frac{9}{16}} = \frac{2}{3}. \end{aligned}$$

This illustrates the equivalence of (3) and (4) in the context of this example.

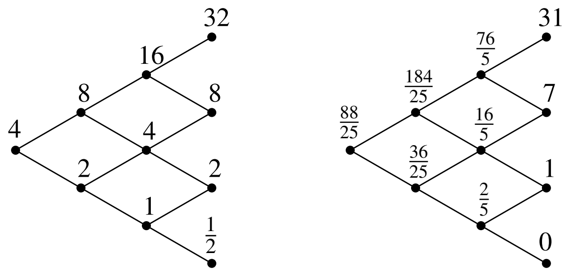

Figure 2. Stock price (left) and option value (right) trees.

Remark 3.5. Although (3) and (4) give us equivalent results in the binomial case, it is important to note that (4) is specifically limited to the binomial case, while (3) can be extended to general discrete-time market models, including incomplete models. In the following section, we apply (3) to the incomplete trinomial model.

4. Incomplete markets

While the binomial model is often used as an introductory tool, most models used in practice exhibit incompleteness, that is, for some securities, it is not possible to construct a hedging strategy that perfectly replicates its payoff. We now introduce a tractable example of an incomplete market.

Trinomial asset pricing model. An example of an incomplete market is based on the *trinomial asset pricing model*. Similar to the binomial model, we have a risky asset and a risk-free asset, and the value of the risky asset at each time step is determined by a small set of outcomes. This time, we have three possible outcomes for the coin flip instead of two. That is, along with the possibility of an increase by a factor of u and decrease by a factor of d, we allow for the possibility that the stock price does not change between two consecutive time steps. See Figure 3.

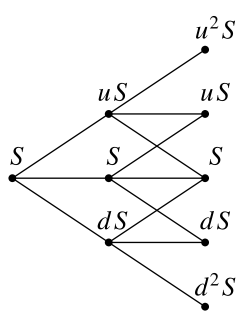

Figure 3. Example of a 2-period trinomial model.

Remark 4.1. For the above formulation of the trinomial model, we again require that $u = 1/d$. However, as in the binomial model, this requirement is not necessary.

Attempting backward recursion on this model using the nondiscounted wealth process, we get

$$\begin{aligned} X_2(\omega_1\omega_2) &= (1+r)(X_1(\omega_1) - \vartheta_1(\omega_1)S_1(\omega_1)) + \vartheta_1(\omega_1)S_2(\omega_1\omega_2),\\ X_1(\omega_1) &= (1+r)(X_0 - \vartheta_0 S_0) + \vartheta_0 S_1(\omega_1). \end{aligned}$$

Note that there are three possible values for ω_1 and three possible values for ω_2, and we must solve for ϑ_0, X_0, each $\vartheta_1(\omega_1)$, and each $X_1(\omega_1)$, giving us eight unknowns and twelve equations. This makes the system overdetermined. In particular, simple matrix calculations reveal that, in general, we have no solution for X_0.

5. Stability under model perturbations

We now turn to the question of stability of the Föllmer–Schweizer decomposition. There are different kinds of perturbations one can consider, and for example, stability with respect to perturbations of V_N is considered in [Monat and Stricker 1995]. In this paper, we consider perturbations of the stock price process. For the stability analysis, as we work on finite probability spaces, the exact form of perturbations is not important, and we will suppose that there is a family of adapted stock price processes parametrized by ε, $(S^\varepsilon)_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ for some $\varepsilon_0 > 0$. An example of such a family corresponds to linear perturbations of the drift and volatility considered in the following section. Here we will only suppose that

$$S^\varepsilon \to S^0$$

in the sense that

$$\lim_{\varepsilon\to 0} S_n^\varepsilon(\omega) = S_n^0(\omega) \quad \text{for every } n \in \{0, \dots, N\} \text{ and } \omega \in \Omega. \tag{8}$$

The following result asserts that the Föllmer–Schweizer decomposition on finite probability spaces is stable under perturbations of the stock of the form (8).

Theorem 5.1. *On a finite probability space, let us consider a family of stock price processes* $((S_n^\varepsilon)_{n\in\{0,\dots,N\}})_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$, *for some* $\varepsilon_0 > 0$, *satisfying* (8). *Let* V_N *be given. Then the corresponding family of the Föllmer–Schweizer decompositions*

$$V_N = V_0^\varepsilon + \sum_{j=1}^N \hat\vartheta_j^\varepsilon \Delta S_j^\varepsilon + L_N^\varepsilon, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0),$$

satisfies

$$\begin{aligned} &\lim_{\varepsilon\to 0} V_0^\varepsilon = V_0^0,\\ &\lim_{\varepsilon\to 0} L_n^\varepsilon = L_n^0, \quad n \in \{0, \dots, N\},\\ &\lim_{\varepsilon\to 0} \hat\vartheta_n^\varepsilon = \hat\vartheta_n^0, \quad n \in \{1, \dots, N\}, \end{aligned} \tag{9}$$

where the equalities hold for every $\omega \in \Omega$. As a consequence, we also have

$$\lim_{\varepsilon \to 0} \sum_{j=1}^{n} \hat{\vartheta}_j^{\varepsilon} \Delta S_j^{\varepsilon} = \sum_{j=1}^{n} \hat{\vartheta}_j^{0} \Delta S_j^{0}, \quad n \in \{1, \dots, N\}, \ \omega \in \Omega. \tag{10}$$

Proof. The proof goes recursively, backward in n. First, let us consider $n = N$. From (8), we get

$$\lim_{\varepsilon \to 0} \Delta S_N^{\varepsilon} = S_N^0, \quad \omega \in \Omega.$$

Further, working on a finite probability space, via the formal definition of conditional expectation, from (8), without any additional assumptions, we get

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N^{\varepsilon}] = \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N^0].$$

As a consequence, we obtain

$$\begin{aligned}
\lim_{\varepsilon \to 0} \hat{\vartheta}_N^{\varepsilon} &= \lim_{\varepsilon \to 0} \frac{\mathrm{Cov}_{\mathcal{F}_{N-1}}[V_N, \Delta S_N^{\varepsilon}]}{\mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S_N^{\varepsilon}]} \\
&= \lim_{\varepsilon \to 0} \frac{\mathbb{E}_{\mathcal{F}_{N-1}}[(V_N - \mathbb{E}_{\mathcal{F}_{N-1}}[V_N])(\Delta S_N^{\varepsilon} - \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N^{\varepsilon}])]}{\mathbb{E}_{\mathcal{F}_{N-1}}[(\Delta S_N^{\varepsilon} - \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N^{\varepsilon}])^2]} \\
&= \frac{\mathbb{E}_{\mathcal{F}_{N-1}}[(V_N - \mathbb{E}_{\mathcal{F}_{N-1}}[V_N]) \lim_{\varepsilon \to 0}(\Delta S_N^{\varepsilon} - \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N^{\varepsilon}])]}{\mathbb{E}_{\mathcal{F}_{N-1}}[\lim_{\varepsilon \to 0}(\Delta S_N^{\varepsilon} - \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N^{\varepsilon}])^2]} \\
&= \frac{\mathbb{E}_{\mathcal{F}_{N-1}}[(V_N - \mathbb{E}_{\mathcal{F}_{N-1}}[V_N])(\lim_{\varepsilon \to 0} \Delta S_N^{\varepsilon} - \lim_{\varepsilon \to 0} \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N^{\varepsilon}])]}{\mathbb{E}_{\mathcal{F}_{N-1}}[(\lim_{\varepsilon \to 0} \Delta S_N^{\varepsilon} - \lim_{\varepsilon \to 0} \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N^{\varepsilon}])^2]} \\
&= \frac{\mathbb{E}_{\mathcal{F}_{N-1}}[(V_N - \mathbb{E}_{\mathcal{F}_{N-1}}[V_N])(\Delta S_N^{0} - \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N^{0}])]}{\mathbb{E}_{\mathcal{F}_{N-1}}[(\Delta S_N^{0} - \mathbb{E}_{\mathcal{F}_{N-1}}[\Delta S_N^{0}])^2]} \\
&= \frac{\mathrm{Cov}_{\mathcal{F}_{N-1}}[V_N, \Delta S_N^{0}]}{\mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S_N^{0}]} \\
&= \hat{\vartheta}_N^{0},
\end{aligned} \tag{11}$$

where the chain of equalities holds for every $\omega \in \Omega$. If $N = 1$, (11) implies the third equality in (9), and the remaining assertions of the theorem follow. If $N > 1$, defining

$$A_n^{\varepsilon} := V_N - \sum_{j=n+1}^{N} \hat{\vartheta}_j^{\varepsilon} \Delta S_j^{\varepsilon}, \quad n \in \{0, \dots, N-1\}, \ \varepsilon \in (-\varepsilon_0, \varepsilon_0),$$

from (11), we get

$$\lim_{\varepsilon \to 0} A_{N-1}^{\varepsilon} = A_{N-1}^{0}, \quad \omega \in \Omega.$$

Consequently, similarly to (11), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\hat{\vartheta}^{\varepsilon}_{N-1} &= \lim_{\varepsilon\to 0}\frac{\operatorname{Cov}_{\mathcal{F}_{N-2}}[A^{\varepsilon}_{N-1},\Delta S^{\varepsilon}_{N-1}]}{\operatorname{Var}_{\mathcal{F}_{N-2}}[\Delta S^{\varepsilon}_{N-1}]}\\ &= \frac{\operatorname{Cov}_{\mathcal{F}_{N-2}}[\lim_{\varepsilon\to 0}A^{\varepsilon}_{N-1},\lim_{\varepsilon\to 0}\Delta S^{\varepsilon}_{N-1}]}{\operatorname{Var}_{\mathcal{F}_{N-2}}[\lim_{\varepsilon\to 0}\Delta S^{\varepsilon}_{N-1}]}\\ &= \frac{\operatorname{Cov}_{\mathcal{F}_{N-2}}[A^{0}_{N-1},\Delta S^{0}_{N-1}]}{\operatorname{Var}_{\mathcal{F}_{N-2}}[\Delta S^{0}_{N-1}]}\\ &= \hat{\vartheta}^{0}_{N-1},\quad \omega\in\Omega.\end{aligned}\tag{12}$$

Proceeding in such a manner, one can show that

$$\lim_{\varepsilon\to 0}\hat{\vartheta}^{\varepsilon}_{n} = \hat{\vartheta}^{0}_{n},\quad n\in\{1,\dots,N\},\ \omega\in\Omega,$$

which is the last equality in (9). In turn, this and (8) imply (10). Therefore, for every $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, by taking expectation in

$$V_N = V^{\varepsilon}_0 + \sum_{j=1}^{N}\hat{\vartheta}^{\varepsilon}_j\Delta S^{\varepsilon}_j + L^{\varepsilon}_N,\tag{13}$$

and using $\mathbb{E}[L^{\varepsilon}_N]=0$, $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, we deduce via (10) that $\lim_{\varepsilon\to 0}V^{\varepsilon}_0 = V^0_0$; i.e., the first equality in (9) holds. Consequently, as the left-hand side in (13) does not depend on ε, from (13), the convergence of V^{ε}_0 to V^0_0 and (10), we deduce that $\lim_{\varepsilon\to 0}L^{\varepsilon}_N = L^0_N$ for every $\omega\in\Omega$. Finally, as L^{ε}'s are $\mathbb{P}$-martingales, using $\mathbb{E}_{\mathcal{F}_n}[L^{\varepsilon}_N]=L^{\varepsilon}_n$, we conclude that

$$\lim_{\varepsilon\to 0}L^{\varepsilon}_n = L^0_n\quad\text{for every } n\in\{0,\dots,N\}\text{ and }\omega\in\Omega,$$

which is the second equality in (9). □

6. Asymptotic analysis

While stability tells us whether or not there is a convergence of the problem outputs under perturbations of the input data, the asymptotic analysis gives a quantitative estimate of how the problem responds to such perturbations. In order to make such estimates, assuming merely $S^{\varepsilon}\to S^0$ as in (8) from the previous section is not enough. We need to parametrize perturbations more precisely. Before stating our form of perturbations, one can also consider that, in practice, the dynamics of the stock price process is commonly decomposed into two parts. The first one is drift, or in discounted settings, it can also be stated as the market price of risk. This part is responsible for the trend of the stock. The second part captures the fluctuations, that is, how much can the stock fluctuate in a given interval. Therefore, one can formulate the following dynamics of the stock for the base model:[2]

$$\Delta S_n^0 = \lambda_n + \sigma_n \Delta W_n = \lambda_n \Delta t + \sigma_n \Delta W_n, \quad n \in \{1, \dots, N\}, \tag{14}$$

where λ and σ are predictable processes, $\Delta t \equiv 1$ represents the change in time, and ΔW_n is an $\mathcal{F}_n$-measurable increment of a martingale with initial value 0, interpreted as an error or "noise" term, where we additionally suppose that the standard deviation of ΔW_n is $\Delta t = 1$ for normalization purposes. Equation (14) is also consistent with the so-called semimartingale decomposition of the stock price process, where a semimartingale can be defined to be a process that can be written as a sum of a martingale (noise term) and an adapted process (drift term). In the finite probability settings, in fact, the drift term in the semimartingale decomposition can be chosen to be predictable.

Remark 6.1. We observe that given any process S, and once the time step $\Delta t > 0$ is fixed (and is constant in this paper, for simplicity of notations), the processes λ and σ can be obtained as follows:

$$\begin{aligned} \lambda_n &= \frac{\mathbb{E}_{\mathcal{F}_{n-1}}[\Delta S_n]}{\Delta t}, \\ \sigma_n &= \sqrt{\mathrm{Var}_{\mathcal{F}_{n-1}}[\Delta S_n]}, \\ \Delta W_n &= \frac{\Delta S_n^0 - \lambda_n \Delta t}{\sigma_n} 1_{\{\sigma_n > 0\}}, \quad n \in \{1, \dots, N\}. \end{aligned}$$

For perturbations of the underlying dynamics in (14), we can consider simultaneous (or separate) distortions of both the drift and the noise terms in (14). Therefore, we now define model perturbations as

$$\Delta S_n^\varepsilon = (\lambda_n + \varepsilon\lambda_n')\Delta t + (\sigma_n + \varepsilon\sigma_n')\Delta W_n + \varepsilon\sigma_n''\Delta W_n^\perp, \quad n \in \{1, \dots, N\}, \ \varepsilon \in (-\varepsilon_0, \varepsilon_0),$$

where λ', σ', and σ'' are predictable processes, $\Delta W_n^\perp$ is an $\mathcal{F}_n$-measurable normalized noise term, which is conditionally uncorrelated from ΔW_n, that is,

$$\begin{aligned} \mathbb{E}_{\mathcal{F}_{n-1}}[\Delta W_n^\perp] &= 0, \\ \mathrm{Var}_{\mathcal{F}_{n-1}}[\Delta W_n^\perp] &= 1, \\ \mathrm{Cov}_{\mathcal{F}_{n-1}}[\Delta W_n, \Delta W_n^\perp] &= 0, \quad n \in \{1, \dots, N\}, \end{aligned}$$

and ε_0 is a strictly positive constant. With

$$\Delta S_n' := \lambda_n' \Delta t + \sigma_n' \Delta W_n + \sigma_n'' \Delta W_n^\perp, \quad n \in \{1, \dots, N\}, \tag{15}$$

we can rewrite the dynamics of the perturbed processes as

$$\Delta S_n^\varepsilon = \Delta S_n^0 + \varepsilon \Delta S_n', \quad n \in \{1, \dots, N\}, \ \varepsilon \in (-\varepsilon_0, \varepsilon_0). \tag{16}$$

[2]Here the base model is the one that corresponds to $\varepsilon = 0$ in the notation of Sections 5 and 6.

Remark 6.2. Similarly to the process ΔS, if one starts from a given perturbation process $\Delta S'$, it can be represented in the form (15) as follows: In the notation of Remark 6.1, we have

$$\lambda'_n = \frac{\mathbb{E}_{\mathcal{F}_{n-1}}[\Delta S'_n]}{\Delta t}, \quad n \in \{1, \dots, N\},$$

and the computation of the remaining parameters goes along the lines of [Shreve 2004b, Example 2.3.3, p. 72]. Let us consider

$$\Delta S'_n - \mathbb{E}_{\mathcal{F}_{n-1}}[\Delta S'_n] = \sigma'_n \Delta W_n + (\Delta S'_n - \mathbb{E}_{\mathcal{F}_{n-1}}[\Delta S'_n] - \sigma'_n \Delta W_n), \quad n \in \{1, \dots, N\},$$

where σ'_n has to be determined in such a way that the term in the brackets is conditionally uncorrelated from ΔW_n. As $\mathrm{Var}_{\mathcal{F}_{n-1}}[\Delta W_n] = 1$, direct calculations give

$$\sigma'_n = \mathbb{E}_{\mathcal{F}_{n-1}}[\Delta W_n, \Delta S'_n - \mathbb{E}_{\mathcal{F}_{n-1}}[\Delta S'_n]] = \mathrm{Cov}_{\mathcal{F}_{n-1}}[\Delta W_n, \Delta S'_n], \quad n \in \{1, \dots, N\}.$$

Therefore, for σ''_n and $\Delta W_n^{\perp}$, we get

$$\begin{aligned}
\sigma''_n &= \sqrt{\mathrm{Var}_{\mathcal{F}_{n-1}}[\Delta S'_n - \mathbb{E}_{\mathcal{F}_{n-1}}[\Delta S'_n] - \sigma'_n \Delta W_n]} = \sqrt{\mathrm{Var}_{\mathcal{F}_{n-1}}[\Delta S'_n - \sigma'_n \Delta W_n]},\\
\Delta W_n^{\perp} &= \frac{\Delta S'_n - \mathbb{E}_{\mathcal{F}_{n-1}}[\Delta S'_n] - \sigma'_n \Delta W_n}{\sigma''_n} 1_{\{\sigma''_n > 0\}}, \quad n \in \{1, \dots, N\}.
\end{aligned}$$

The following theorem gives the leading-order correction terms to the components of the Föllmer–Schweizer decomposition under perturbations of the form (16). Let us consider the associated family of the Föllmer–Schweizer decompositions

$$V_N = V_0^{\varepsilon} + \sum_{j=1}^{N} \hat{\vartheta}_j^{\varepsilon} \Delta S_j^{\varepsilon} + L_N^{\varepsilon}, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0), \tag{17}$$

and let us define recursively, backward in n, the process $\hat{\vartheta}'$, which will be proven in Theorem 6.3 to be the first-order correction to the optimal strategy, as

$$\begin{aligned}
\hat{\vartheta}'_N &:= \frac{\mathrm{Cov}_{\mathcal{F}_{N-1}}[V_N, \Delta S'_N]}{\mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S_N^0]} - 2\frac{\mathrm{Cov}_{\mathcal{F}_{N-1}}[V_N, \Delta S_N^0]\,\mathrm{Cov}_{\mathcal{F}_{N-1}}[\Delta S_N^0, \Delta S'_N]}{(\mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S_N^0])^2},\\
\hat{\vartheta}'_n &:= \frac{1}{\mathrm{Var}_{\mathcal{F}_{n-1}}[\Delta S_n^0]}\Bigg(\mathrm{Cov}_{\mathcal{F}_{n-1}}\Bigg[V_N - \sum_{j=n+1}^{N} \hat{\vartheta}_j^0 \Delta S_j^0, \Delta S'_n\Bigg]\\
&\qquad\qquad - \mathrm{Cov}_{\mathcal{F}_{n-1}}\Bigg[\sum_{j=n+1}^{N} \hat{\vartheta}'_j \Delta S_j^0 + \sum_{j=n+1}^{N} \hat{\vartheta}_j^0 \Delta S'_j, \Delta S_n^0\Bigg]\Bigg)\\
&\quad - 2\frac{\mathrm{Cov}_{\mathcal{F}_{n-1}}[V_N - \sum_{j=n+1}^{N} \hat{\vartheta}_j^0 \Delta S_j^0, \Delta S_n^0]\,\mathrm{Cov}_{\mathcal{F}_{N-1}}[\Delta S_n^0, \Delta S'_n]}{(\mathrm{Var}_{\mathcal{F}_{n-1}}[\Delta S_n^0])^2},
\end{aligned} \tag{18}$$

where $n \in \{N-1, \dots, 1\}$.

Theorem 6.3. *On a finite probability space, let us consider a family of stock price processes* $((S^\varepsilon_n)_{n\in\{0,\dots,N\}})_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ *for some* $\varepsilon_0 > 0$, *where the increments*, $\Delta S^{\varepsilon\prime}_n$, *are given via* (16). *Let* V_N *be given. Then the components of the family of the Föllmer–Schweizer decompositions defined in* (17) *satisfy*

$$\lim_{\varepsilon\to 0}\frac{V^\varepsilon_0 - V^0_0}{\varepsilon} = -\mathbb{E}\Bigg[\sum_{j=1}^{N}\hat{\vartheta}^0_j\Delta S'_j + \sum_{j=1}^{N}\hat{\vartheta}'_j\Delta S^0_j\Bigg],$$

$$\lim_{\varepsilon\to 0}\frac{\hat{\vartheta}^\varepsilon_n - \hat{\vartheta}^0_n}{\varepsilon} = \hat{\vartheta}'_n, \quad n\in\{1,\dots,N\},\ \omega\in\Omega,$$

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{L^\varepsilon_n - L^0_n}{\varepsilon} \\ &\quad= -\mathbb{E}_{\mathcal{F}_n}\Bigg[\sum_{j=1}^{N}\hat{\vartheta}'_j\Delta S^0_j - \mathbb{E}\Bigg[\sum_{j=1}^{N}\hat{\vartheta}'_j\Delta S^0_j\Bigg]\Bigg] - \mathbb{E}_{\mathcal{F}_n}\Bigg[\sum_{j=1}^{N}\hat{\vartheta}^0_j\Delta S'_j - \mathbb{E}\Bigg[\sum_{j=1}^{N}\hat{\vartheta}^0_j\Delta S'_j\Bigg]\Bigg], \\ &\qquad\qquad n\in\{0,\dots,N\},\ \omega\in\Omega,\end{aligned} \tag{19}$$

where ϑ' *is defined in* (18). *We also have*

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{\sum_{j=1}^{n}\hat{\vartheta}^\varepsilon_j\Delta S^\varepsilon_j - \sum_{j=1}^{n}\hat{\vartheta}^0_j\Delta S^0_j}{\varepsilon} \\ &\qquad= \sum_{j=1}^{n}\hat{\vartheta}^0_j\Delta S'_j + \sum_{j=1}^{n}\hat{\vartheta}'_j\Delta S^0_j, \quad n\in\{1,\dots,N\},\ \omega\in\Omega.\end{aligned} \tag{20}$$

Proof. We will investigate

$$\lim_{\varepsilon\to 0}\frac{\hat{\vartheta}^\varepsilon_n - \hat{\vartheta}^0_n}{\varepsilon}$$

first. For $n = N$, we have

$$\lim_{\varepsilon\to 0}\frac{\hat{\vartheta}^\varepsilon_n - \hat{\vartheta}^0_n}{\varepsilon} = \lim_{\varepsilon\to 0}\Bigg(\frac{\mathrm{Cov}_{\mathcal{F}_{N-1}}[V_N,\Delta S^\varepsilon_N]}{\mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S^\varepsilon_N]} - \frac{\mathrm{Cov}_{\mathcal{F}_{N-1}}[V_N,\Delta S^\varepsilon_N]}{\mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S^\varepsilon_N]}\Bigg)\frac{1}{\varepsilon}. \tag{21}$$

Using the definition of conditional expectation, and Theorem 5.1,[3] we get

$$\lim_{\varepsilon\to 0}\frac{\mathrm{Cov}_{\mathcal{F}_{N-1}}[V_n,\Delta S^\varepsilon_N] - \mathrm{Cov}_{\mathcal{F}_{N-1}}[V_n,\Delta S^0_N]}{\varepsilon} = \mathrm{Cov}_{\mathcal{F}_{N-1}}[V_N,\Delta S'_N].$$

Similarly, we deduce that

$$\lim_{\varepsilon\to 0}\frac{\mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S^\varepsilon_N] - \mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S^0_N]}{\varepsilon} = 2\,\mathrm{Cov}_{\mathcal{F}_{N-1}}[\Delta S^0_N,\Delta S'_N].$$

[3] Below, we apply the assertions of Theorem 5.1 at a family of points ε near the origin; this however causes no difficulty by relabeling S^ε's.

Therefore, in (21), we obtain

$$\lim_{\varepsilon\to 0}\frac{\hat{\vartheta}_n^{\varepsilon}-\hat{\vartheta}_n^{0}}{\varepsilon} = \frac{\mathrm{Cov}_{\mathcal{F}_{N-1}}[V_N,\Delta S_N']\,\mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S_N^0]-2\,\mathrm{Cov}_{\mathcal{F}_{N-1}}[V_N,\Delta S_N^0]\,\mathrm{Cov}_{\mathcal{F}_{N-1}}[\Delta S_N^0,\Delta S_N']}{(\mathrm{Var}_{\mathcal{F}_{N-1}}[\Delta S_N^0])^2},$$

which is precisely $\hat{\vartheta}_N'$. If $N=1$, this completes the proof for

$$\lim_{\varepsilon\to 0}\frac{\hat{\vartheta}_n^{\varepsilon}-\hat{\vartheta}_n^{0}}{\varepsilon}=\vartheta_n'$$

for every n. If $N>1$, defining

$$A_n^{\varepsilon}:=V_N-\sum_{j=n+1}^{N}\hat{\vartheta}_j^{\varepsilon}\Delta S_j^{\varepsilon},\quad n\in\{0,\dots,N-1\},\ \varepsilon\in(-\varepsilon_0,\varepsilon_0),$$

and using Theorem 5.1, we get recursively, backward in n, the following chain of equalities. First, for $n=N-1$, we obtain

$$\lim_{\varepsilon\to 0}\frac{A_n^{\varepsilon}-A_n^{0}}{\varepsilon}=-\sum_{j=n+1}^{N}\hat{\vartheta}_j'\Delta S_j^0-\sum_{j=n+1}^{N}\hat{\vartheta}_j^0\Delta S_j'.$$

Therefore, using Theorem 5.1 again, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{\mathrm{Cov}_{\mathcal{F}_{n-1}}[A_n^{\varepsilon},\Delta S_n^{\varepsilon}]-\mathrm{Cov}_{\mathcal{F}_{n-1}}[A_n^{0},\Delta S_n^{0}]}{\varepsilon}\\
&=\mathrm{Cov}_{\mathcal{F}_{n-1}}\Bigg[-\sum_{j=n+1}^{N}\hat{\vartheta}_j'\Delta S_j^0-\sum_{j=n+1}^{N}\hat{\vartheta}_j^0\Delta S_j',\Delta S_n^0\Bigg]+\mathrm{Cov}_{\mathcal{F}_{n-1}}[A_n^0,\Delta S_n']\\
&=\mathrm{Cov}_{\mathcal{F}_{n-1}}\Bigg[-\sum_{j=n+1}^{N}\hat{\vartheta}_j'\Delta S_j^0-\sum_{j=n+1}^{N}\hat{\vartheta}_j^0\Delta S_j',\Delta S_n^0\Bigg]+\mathrm{Cov}_{\mathcal{F}_{n-1}}\Bigg[V_N-\sum_{j=n+1}^{N}\hat{\vartheta}_j^0\Delta S_j^0,\Delta S_n'\Bigg],
\end{aligned}$$

and thus, in (21), we conclude that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\hat{\vartheta}_n^{\varepsilon}-\hat{\vartheta}_n^{0}}{\varepsilon}&=\frac{1}{\mathrm{Var}_{\mathcal{F}_{n-1}}[\Delta S_n^0]}\Bigg(\mathrm{Cov}_{\mathcal{F}_{n-1}}\Bigg[V_N-\sum_{j=n+1}^{N}\hat{\vartheta}_j^0\Delta S_j^0,\Delta S_n'\Bigg]\\
&\qquad-\mathrm{Cov}_{\mathcal{F}_{n-1}}\Bigg[\sum_{j=n+1}^{N}\hat{\vartheta}_j'\Delta S_j^0+\sum_{j=n+1}^{N}\hat{\vartheta}_j^0\Delta S_j',\Delta S_n^0\Bigg]\Bigg)\\
&\quad-\frac{2\,\mathrm{Cov}_{\mathcal{F}_{n-1}}\big[V_N-\sum_{j=n+1}^{N}\hat{\vartheta}_j^0\Delta S_j^0,\Delta S_n^0\big]\,\mathrm{Cov}_{\mathcal{F}_{N-1}}[\Delta S_n^0,\Delta S_n']}{(\mathrm{Var}_{\mathcal{F}_{n-1}}[\Delta S_n^0])^2},
\end{aligned}$$

which is exactly $\hat{\vartheta}'_n$ from (18) for $n = N-1$. Proceeding this way, we can establish (18) for every $n \geq 1$. This completes the proof of the third equality in (19). Now, (20) follows from the third equality in (19) and Theorem 5.1.

To establish the first two equalities in (19), we proceed as follows. For every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, taking the expectation in (17) and observing that the left-hand side does not depend on ε, as well as that $\mathbb{E}[L^\varepsilon_N] = 0$, we get

$$V^\varepsilon_0 + \mathbb{E}\Bigg[\sum_{j=1}^N \hat{\vartheta}^\varepsilon_j \Delta S^\varepsilon_j\Bigg] = V^0_0 + \mathbb{E}\Bigg[\sum_{j=1}^N \hat{\vartheta}^0_j \Delta S^0_j\Bigg], \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

Collecting the terms and dividing by $\varepsilon \neq 0$, we obtain

$$\frac{V^\varepsilon_0 - V^0_0}{\varepsilon} = \frac{\mathbb{E}\big[\sum_{j=1}^N \hat{\vartheta}^0_j \Delta S^0_j - \sum_{j=1}^N \hat{\vartheta}^\varepsilon_j \Delta S^\varepsilon_j\big]}{\varepsilon}, \quad \varepsilon \in (-\varepsilon_0, 0) \cup (0, \varepsilon_0).$$

Taking the limit as $\varepsilon \to 0$, and using (20), we conclude that

$$\lim_{\varepsilon \to 0} \frac{V^\varepsilon_0 - V^0_0}{\varepsilon} = -\mathbb{E}\Bigg[\sum_{j=1}^N \hat{\vartheta}^0_j \Delta S'_j + \sum_{j=1}^N \hat{\vartheta}'_j \Delta S^0_j\Bigg],$$

which is precisely the first equality in (19).

To obtain the remaining assertion in (19), we observe that from (17), (20) and the other two assertions in (19), we immediately obtain

$$\begin{aligned}
\lim_{\varepsilon \to 0} \frac{L^\varepsilon_N - L^0_N}{\varepsilon} &= -\lim_{\varepsilon \to 0} \frac{V^\varepsilon_0 - V^0_0}{\varepsilon} - \lim_{\varepsilon \to 0} \frac{\sum_{j=1}^N \hat{\vartheta}^\varepsilon_j \Delta S^\varepsilon_j - \sum_{j=1}^N \hat{\vartheta}^0_j \Delta S^0_j}{\varepsilon} \\
&= \mathbb{E}\Bigg[\sum_{j=1}^N \hat{\vartheta}^0_j \Delta S'_j + \sum_{j=1}^N \hat{\vartheta}'_j \Delta S^0_j\Bigg] - \sum_{j=1}^N \hat{\vartheta}^0_j \Delta S'_j - \sum_{j=1}^N \hat{\vartheta}'_j \Delta S^0_j \\
&= -\Bigg(\sum_{j=1}^N \hat{\vartheta}^0_j \Delta S'_j - \mathbb{E}\Bigg[\sum_{j=1}^N \hat{\vartheta}^0_j \Delta S'_j\Bigg]\Bigg) - \Bigg(\sum_{j=1}^N \hat{\vartheta}'_j \Delta S^0_j - \mathbb{E}\Bigg[\sum_{j=1}^N \hat{\vartheta}'_j \Delta S^0_j\Bigg]\Bigg).
\end{aligned}$$

If $N = 1$, this completes the proof. If $N > 1$, for $n < N$, we have $\mathbb{E}_{\mathcal{F}_n}[L^\varepsilon_N] = L^\varepsilon_n$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Therefore, using (17) and taking the conditional expectation, we obtain

$$\begin{aligned}
\frac{L^\varepsilon_n - L^0_n}{\varepsilon} &= \frac{\mathbb{E}_{\mathcal{F}_n}[L^\varepsilon_N - L^0_N]}{\varepsilon} \\
&= -\frac{V^\varepsilon_0 - V^0_0}{\varepsilon} - \frac{\mathbb{E}_{\mathcal{F}_n}\big[\sum_{j=1}^N \hat{\vartheta}^\varepsilon_j \Delta S^\varepsilon_j - \sum_{j=1}^N \hat{\vartheta}^0_j \Delta S^0_j\big]}{\varepsilon}.
\end{aligned}$$

Taking the limit, and using (20) and the first equality in (19), we conclude that

$$\lim_{\varepsilon\to 0}\frac{L_n^\varepsilon - L_n^0}{\varepsilon} = -\mathbb{E}_{\mathcal{F}_n}\Bigg[\sum_{j=1}^{N}\hat{\vartheta}_j'\Delta S_j^0 - \mathbb{E}\Bigg[\sum_{j=1}^{N}\hat{\vartheta}_j'\Delta S_j^0\Bigg]\Bigg] - \mathbb{E}_{\mathcal{F}_n}\Bigg[\sum_{j=1}^{N}\hat{\vartheta}_j^0\Delta S_j' - \mathbb{E}\Bigg[\sum_{j=1}^{N}\hat{\vartheta}_j^0\Delta S_j'\Bigg]\Bigg].$$

This completes the proof of the theorem. □

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