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# Combinatorial random knots

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We explore free knot diagrams, which are projections of knots into the plane which don't record over/under data at crossings. We consider the combinatorial question of which free knot diagrams give which knots and with what probability. Every free knot diagram is proven to produce trefoil knots, and certain simple families of free knots are completely worked out. We make some conjectures (supported by computer-generated data) about bounds on the probability of a knot arising from a fixed free diagram being the unknot, trefoil, or figure-eight knot.

## 1. Introduction

Knots and links have been objects of mathematical interest for centuries. In 1833, Gauss found the linking integral for two loops. Knots and links were some of the first objects to be studied topologically. They remain a cornerstone of the field of topology, and are useful in many settings beyond their innate two-/three-dimensionalness.

The combinatorial study of knots dates back to Reidemeister, who described a set of three moves which generate all equivalences of knot diagrams. In this article, we investigate a combinatorial aspect of knot theory coming from considering knot diagrams without crossing data, and the knots that result from random assignment of crossing data. Suppose we have a "free knot diagram", as shown in Figure 1. If we randomly assign over/under data to each crossing, how many unknots will we get? For a generic free knot diagram, will any assignment of crossings produce trefoils, figure-eight knots, the knot $5_2$, etc.? What is the average crossing number of all knots produced by such assignments to a given free knot diagram?

We began this project by taking an experimental approach: For a set of free knot diagrams with a smallish number of crossings, we computed all the knots which result from assignments of crossing data. We did this using a Mathematica

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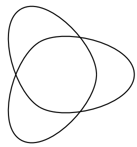


**Figure 1.** A free knot diagram.

program to calculate the Jones polynomial: though not a perfect invariant, the Jones polynomial can tell apart all prime knots with nine or fewer crossings.

After looking at the data, we made some observations, and in this article prove many of them. As a starting point, we have the following:

**Theorem 1.1.** *A random knot coming from an $n$-crossing knot diagram has probability at least $2n/2^n$ of being the unknot.*

This theorem sets a lower bound on the possible amount of unknots, which is intriguing, but does not tell us much information about the vast majority of knots. For example, once we start looking at diagrams with six crossings, this theorem only describes one third of all produced unknots. We want to establish stronger bounds.

Another observation we quickly made was the following:

**Theorem 1.2.** *Let $K$ be a free knot diagram which is nontrivial. Then $K$ has some assignment of crossings that produces a trefoil diagram.*

We examine four particular categories of free knot diagrams, and describe the knots that result. Computational evidence suggests that some of these knots realize upper or lower bounds on unknots, trefoils, or figure-eight knots.

**Conjecture 1.3.** *The free 2-braid knot diagram with $n$ crossings realizes the upper bound on trefoils for all sufficiently complicated free diagrams with $n$ and $n+1$ crossings.*

**Conjecture 1.4.** *The free $(2, n)$-diagram in Figure 2 realizes the lower bound on the trefoils and the upper bound on the unknots for all sufficiently complicated free diagrams with $n+2$ crossings.*

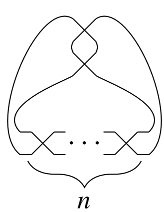



**Figure 2.** A free $(2, n)$-diagram.

Based on the determined formula for unknots produced by a randomization of a free $(2, n)$-knot, we also propose an absolute maximum of resultant unknot probability coming from a nontrivial free knot diagram, of 0.75.

The structure of this article is as follows. Section 2 establishes notation and defines free knot diagrams. Our general results are in Section 3. The most interesting of these is that every free knot diagram produces trefoils. Sections 4, 5, and 6 compute the complete resolution of 2-braid knot diagrams, the $(2, n)$-tangle knots, and the $(2, 1, n)$-tangle knots, with Section 5 including discussion of the more general $(k, n)$-tangle knots.

At the end of each section, we include some conjectures which are supported by the data we generated, but do not seem accessible to prove at the moment. Future directions to investigate include these conjectures and working out resolutions of other knot families. It would be great to develop a more theoretical understanding of the role that various structures play in knot resolutions: understanding tangle structure and its role in generating (apparently) minimal-unknot and maximal-unknot examples, and understanding braid structure and proving that the figure-eight knot is universal among prime knot diagrams with braid index 3 and higher.

We thank the referee for calling our attention to [Henrich et al. 2013], which resolves the 2-braid knot diagrams (our Theorem 4.4) and $(2, n)$-tangle knots (Theorem 5.4), and [Cantarella et al. 2017], which, using an immediate consequence of [Taniyama 1989] confirms the existence of the trefoil (Theorem 3.5), figure-eight (Observation 4.9), and the knot $5_2$ as resultants of all sufficiently complicated free knot diagrams. We also note [Medina and Salazar 2019; Medina et al. 2019], which contribute a much more systematic understanding of particular knots which must result from free knots. Our discussion of conjectured bounds on unknot, trefoil, and figure-eight probabilities based on the free closures of rational tangles and free composite knots, in addition to our direct proof of Theorem 3.5, appear unique to this work.

## 2. Background

A knot is a smooth embedding of the circle $\mathbb{S}^1$ into Euclidean three-space $\mathbb{R}^3$, considered up to isotopy. The mathematical study of knots, however, quickly turns into a study of essentially two-dimensional objects as shown in Figure 3. This is a

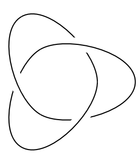


**Figure 3.** A knot diagram.

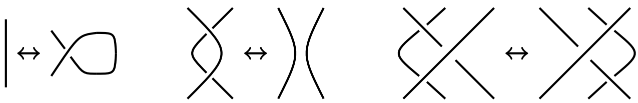



**Figure 4.** The Reidemeister moves.

*knot diagram*: a smooth projection of a knot into $\mathbb{R}^2$ in which all crossings have exactly two transverse (not tangent) strands, and the (barely) three-dimensional data of which strand crosses above is denoted by a break in the understand. A *strand* of a knot is the image under the embedding of any interval of the original circle. A *link* is the multicomponent generalization of a knot.

Given two knot diagrams, how do we know if they represent the same knot? We may attempt to directly manipulate one diagram into another. Reidemeister moves, shown in Figure 4, are three isotopies between diagrams which generate all isotopies.

A simple-to-define knot invariant is the *unknotting number*. It is the least number of crossings one can change in any knot diagram of the given knot to unknot it. This change is shown in Figure 5, where the "overstrand" becomes the "understrand" and vice versa. While straightforward to compute for diagrams, the difficulty arises when attempting to show that no other projection of the knot can be unknotted with fewer changes.

The question we consider in this article, of which knots come from which free knot diagrams, is in some sense a converse to the question of unknotting number. Instead of starting with a knot diagram and trying to change it to reach the unknot, we start with only the shape of a knot diagram, and ask where we may land by assigning its crossings.

Random knots have been studied before, for example from a geometric point of view relating to their appearance in random walks and polymers [Dobay et al. 2003]. Some information is known about their average writhe [Diao et al. 2010].

In order to investigate the combinatorial properties of "crossingless" knots, we define a free knot diagram, meaning a projection into the plane of a knot which does not record which strands go over which others at crossings. It is more convenient, however, to say the following:

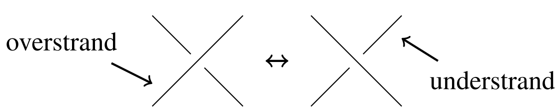
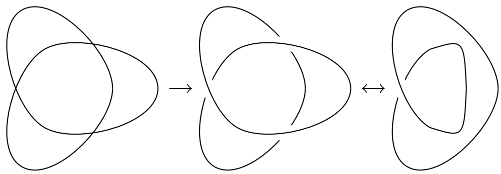



**Figure 5.** Changing a crossing.

**Definition 2.1.** A *free knot diagram* is a planar 4-valent graph, considered up to planar isotopy (i.e., continuous deformations in the plane which preserve vertices and edges). The 4-valent vertices are called *free crossings*.

Free knot diagrams have been studied before in [Manturov and Manturov 2010; Manturov 2012; Hanaki 2010] and in [Henrich et al. 2011a; 2011b], where free knot diagrams are called knot shadows (and mixed diagrams, defined below, are called pseudodiagrams).

**Example 2.2.** Below is a free knot diagram, and an assignment of its crossings which produces the unknot:



Similar simplification with different assignments shows the free trefoil diagram's eight resultants are six unknots and two trefoils.

We may produce a free knot diagram from any knot diagram, by forgetting the over/under information of the crossings:

**Definition 2.3.** The *shape* of a knot diagram is the free knot diagram which results from making all the crossings of the original knot diagram into free crossings.

Reidemeister moves do not apply to free knot diagrams, since planar isotopies of free knot diagrams preserve crossings. A Reidemeister move (incorrectly) applied to a free knot diagram would create a distinct free knot diagram. We can move back to more familiar ground by "assigning" crossings:

**Definition 2.4.** When we *assign* a free crossing, we choose an overstrand and an understrand at that crossing. An *assignment* of a free knot is a choice of how to assign each crossing.

**Definition 2.5.** A *mixed knot diagram* is the projection into the plane of a knot, which records over/under information for only some crossings.

In a mixed knot diagram, we may freely apply Reidemeister moves whenever they involve genuine crossings and no free crossings.

To answer the question of which knots a free knot diagram produces, we assign every possible combination of crossings on the free knot diagram. The two choices at each crossing ensure that for a free knot diagram with $n$ crossings, there are $2^n$ total resultants. The computation of all resultants creates a sample space of outcomes for a random assignment of crossings for a free knot diagram.

**Definition 2.6.** The *resultant knot probability* for a given knot $R$ from a free knot diagram $F$ is the probability that the randomization process applied to $F$ will produce $R$.

Whenever a resultant knot is produced, its mirror image is also produced. Thus, we do not distinguish knots and their mirror images, and the minimum resultant knot probability for an $n$-crossing free knot diagram is $2/2^n$.

The direct calculation of resultant knot probabilities is intensive, and computation time increases quickly as $n$ increases. In the online supplement, we list the probabilities of the unknot, trefoil, and figure-eight knots, in addition to the expectation value, for the knot shapes of the knots with at most nine crossings. The question of what bounds we can place on resultants in general is explored in the next section.

## 3. General results

It is well known that any knot diagram can have some of its crossings changed to represent the unknot. The algorithm below for doing so is also well-known; see, e.g., [Adams 2004]. We include this proof as a warm up, and because it uses techniques we will draw on later.

**Theorem 3.1.** *An $n$-crossing free knot diagram has a minimum of $2n$ assignments which are the unknot.*

*Proof.* By an "arc" of a free knot diagram, we mean a portion of the string containing no crossings (that is, a subset which is homeomorphic to an interval).

Choose an arc and an endpoint, and make that point the highest point of the knot (which we now think of as sitting above the page). Now travel away from the arc, and force our path to travel downhill, by making the current strand an overstrand at every unassigned crossing we encounter. When we arrive at the other endpoint of our initial arc, we are at is the lowest point of the knot. We call the arc containing our initial point the "climb"; the rest of the knot is the "downramp".

Currently we are looking down on the knot from above; rotate the knot so that we view it from the side. From here, we see that the knot we created this way is isotopic to the unknot: since every height, other than the very top and bottom points, has exactly two points at that height (one from the climb, the other from the downramp), the identification with a circle is straightforward.

An $n$-crossing free knot diagram has $2n$ distinct arcs, and as each arc has two endpoints, we have $4n$ distinct climb/downramp pairs on the free knot diagram. But this does not mean we have $4n$ combinatorially distinct unknot diagrams! An easy way to tell apart the unknot diagrams is by their "top track": this is the longest strand containing the climb and only overcrossings. In other words, it stops just before passing through any crossing for the second time. An unknot diagram may

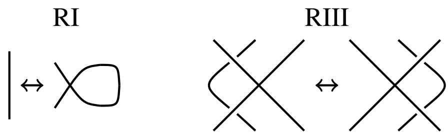



**Figure 6.** The Reidemeister moves of Theorem 3.2.

share its top track with at most one other unknot diagram — this diagram would traverse the same top track in the opposite direction.

Thus, at least $4n/2 = 2n$ of the $2^n$ different knot diagrams associated to an $n$-crossing free knot diagram are diagrams of the unknot. □

**Theorem 3.2.** *Resultant knot probability for a free or mixed knot diagram is invariant under the free first Reidemeister move and a specific mixed third Reidemeister move* (*shown in Figure 6*).

*Proof.* These relations hold for either assignment of the free crossing; thus their use does not change resultant knot probability. □

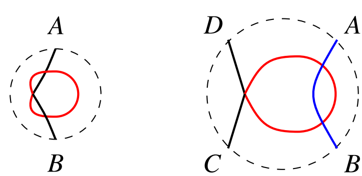
**Definition 3.3.** A *loop* in a knot diagram or a free knot diagram is a segment of the diagram which starts and ends at the same crossing, and does not cross itself otherwise. The *length* of a loop is the number of crossings that segment passes through (with the crossing of origin/terminus only counting once.) All lengths of loops in knot diagrams are odd numbers.

In later theorems, these results can create essentially trivial counterexamples. For example, when discussing upper bounds, the trefoil, whose unknot probability is 0.75, could have $n-3$ loops of length 1 introduced, thus creating a diagram with $n$ crossings and unknot probability 0.75. To prevent these problems, we define the following:

**Definition 3.4.** A *minimal* free knot diagram of $n$ crossings is a free knot diagram which contains no loops of length 1 (i.e., those removable via Reidemeister I).

**Theorem 3.5.** *Let $K$ be a minimal free knot diagram with three or more crossings. Then $K$ has some assignment of crossings making the trefoil knot.*

*Proof.* Suppose $K$ has a loop of length 3. Zoom in on this loop (drawn in red), which has one of two possible forms:

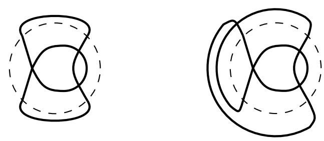


In both cases we assign crossings so that outside of the highlighted disks, the strands of the knot are unknotted. In the first case, along the outside strand, make $A$ the high point, decreasing monotonically to the low point $B$. At any self-crossing, assign the higher strand to go over the lower strand.

Once $A$ and $B$ are externally trivially connected, it is straightforward to assign crossings to get the trefoil knot: traveling within the disk along the strand from $A$, assign the crossings so the current strand goes over, then under, then over. The result is a trefoil knot.

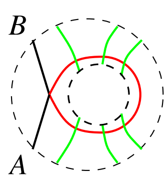
In the second case, there are two subcases to consider. Outside of the highlighted disk, $A$ can either connect to $D$ or $C$. (If $A$ connected to $B$, we would have a link, not a knot.) If $A$ connects to $D$, then we unknot the external strands from each other and themselves by making the $A$-$D$ and $B$-$C$ strands downramps and assigning blended crossings so that $A$-$D$ always goes above $B$-$C$. Similarly, if $A$ connects to $C$, make the $A$-$C$ and $B$-$D$ strands downramps, and assign the blended crossings so that $A$-$C$ always goes above $B$-$D$.

Now, it is straightforward to separate and untangle the external strands, so that the knot looks like one of these:



Both are easily made into the trefoil: For the first one, starting at 12 o'clock on the external strand and traveling clockwise, make that strand pass over, then under, then over the other strands it encounters. For the second one, again starting at 12 o'clock on the external strand and traveling clockwise, make that strand pass over, then under, then under, then over the other strands it encounters.

Now suppose $K$ only has loops of length greater than 3. By induction on the length of a loop, we will show that any knot has a resolution that is the trefoil. Zoom in on a single loop:




The inductive step is to assign crossings and perform isotopies to shorten the length of this loop by two. Do this by picking any green strand at the boundary, declaring that location the high point of a downramp which goes through the highlighted disk, and traveling along that strand, assigning every free crossing encountered to have the chosen green strand go over the other strand, until the strand leaves the disk.

The strand thus assigned passes above all other strands on the interior of the disk. Thus, by a combination of Reidemeister II and mixed Reidemeister III moves, it can be isotoped away from the loop, shortening the length of the loop by 2.

In this way, any free knot diagram with a loop of length more than 3 has some mixed knot in its family with a loop of length 3, thus producing a trefoil knot. □

In the following results, we consider connected sums of free knot diagrams. As connected sum is an operation on knots, and its well-definedness on knot diagrams relies on the Reidemeister moves, we shouldn't expect it to be well-defined on free knots. However, if we are concerned with which knots arise as assignments of a given free knot, then there is no problem: once crossings are assigned, Reidemeister moves are allowed. Thus, in Theorem 3.6 through Consequence 3.17, we simply refer to a connected sum without worrying about which one is meant.

**Theorem 3.6.** *Suppose $K_1$ and $K_2$ are knots with shapes $S_1$ and $S_2$ that are components of a connected sum $S_1\#S_2$. Then the probability of getting an unknot resultant $U$ from $S_1\#S_2$ is $P(S_1 \to U)P(S_2 \to U)$.*

*Proof.* The unknot is a prime knot and can only be created by the knot sum of two unknots; thus $S_1$ and $S_2$ must be simultaneously, and independently, assigned to produce the unknot. □

This simple product of probabilities ensures composites must respect any upper bound on unknot probability that its components are subject to.

**Theorem 3.7.** *Suppose $K_1$ and $K_2$ are knots with shapes $S_1$ and $S_2$ that are components of a connected sum $S_1\#S_2$. Then the probability of getting a resultant nontrivial prime $K_3$ from $S_1\#S_2$ is*

$$P(S_1 \to U)P(S_2 \to K_3) + P(S_1 \to K_3)P(S_2 \to U).$$

*Proof.* If we assign the crossings of $S_2$ such that it becomes the unknot, then we are left with the free knot diagram of $S_1$, from which we can produce $K_3$. Thus, the probability of $K_3$ is at least

$$P(S_1 \to U)P(S_2 \to K_3) + P(S_1 \to K_3)P(S_2 \to U).$$

Further $K_3$-knots could only arise if some connected sum of $K_1$ and $K_2$ or any of their resultants could create $K_3$. The assumption that $K_3$ is prime makes this impossible, so the earlier sum calculates the resultant knot probability for $K_3$. □

**Theorem 3.8.** *Suppose $K_1$ and $K_2$ are prime knots with shapes $S_1$ and $S_2$ that are components of a connected sum $S_1\#S_2$. Then the probability of getting a resultant composite knot $K_3\#K_4$ from $S_1\#S_2$ is*

$$P(S_1 \to U)P(S_2 \to K_3\#K_4) + P(S_1 \to K_3\#K_4)P(S_2 \to U) \\ + P(S_1 \to K_3)P(S_2 \to K_4) + P(S_1 \to K_4)P(S_2 \to K_3).$$

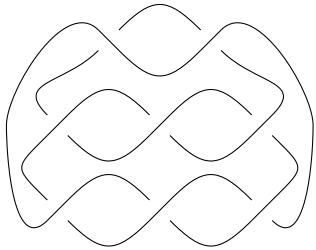


**Figure 7.** The knot $8_5$.

*Proof.* This is a straightforward probability computation. It is worth mentioning that it *is* possible to get composite resultants from shapes of prime knots; for example, the connect sum of two trefoils arises from the shape of knot $8_5$, shown in Figure 7. □

**Definition 3.9.** A *recursive sum* $S^N$ of a knot $K$ with shape $S$ is a connected sum of $N$ copies of $S$.

**Theorem 3.10.** *Let $K_1$ be a prime knot with a shape $S$. Then the resultant knot probability for a nontrivial prime $K_2$, $P(S^{N+1} \to K_2)$, is*

$$P(S \to K_2)P(S \to U)^N + P(S \to U)P(S^N \to K_2).$$

*Proof.* Given $S^{N+1} = S^N \# S$, we apply the formula from Theorem 3.7 and get

$$P(S^N \to U)P(S \to K_2) + P(S^N \to K_2)P(S \to U).$$

Applying Theorem 3.6 $N$ times shows $P(S^N \to U) = P(S \to U)^N$. □

**Definition 3.11.** Using the notation $x_K = P(S^K \to K_2)$, $\alpha = P(S \to U)$, and $\beta = P(S \to K_2)$, we rewrite the recurrence relation for $P(S^{N+1} \to K_2)$ as

$$x_{N+1} = \alpha^N \beta + \alpha x_N.$$

The appropriate initial condition, for a generic shape $S$ and resultant $K$, is $P(S^0 \to K) = 0$, as a crossingless loop, the identity of the connect sum operation, has no nontrivial resultants.

**Theorem 3.12.** *Let $K_1$ be a prime knot with shape $S$ such that a prime $K_2$ is a resultant ($\beta \neq 0$, and thus $\alpha < 1$). Then the sequence of resultant knot probability of $K_2$ from the recursive sum of $S$, $P(S^N \to K_2)$, strictly increases, then has a maximum value at two or fewer steps after which the probability strictly decreases for every additional connected sum.*

*Proof.* Suppose that for some $N$, $x_N \geq x_{N+1}$. Then let us compare $x_{N+2}$ and $x_{N+1}$:

$$x_{N+2} = \alpha^{N+1}\beta + \alpha x_{N+1} < \alpha^N \beta + \alpha x_N = x_{N+1},$$

with the inequality following from $\alpha < 1$ and $x_{N+1} \leq x_N$. Therefore, it is possible that the sequence increases for initial values, but once equality or a decrease is achieved, the sequence strictly decreases for all further indices. □

**Corollary 3.13.** *Let $K_1$ be a prime knot with shape $S$. Then the resultant knot probability of $K_2$ from the recursive sum of $S$, $P(S^N \to K_2)$, has a maximum value at $N = 1$ if and only if $P(S \to U) \leq 0.5$.*

*Proof.* As $x_1 = \beta$ and $x_2 = \alpha^1\beta + \alpha x_1 = 2\alpha\beta$, we know $x_1 \geq x_2$ if and only if $\frac{1}{2} \geq \alpha$. According to Theorem 3.12, $x_1$ is then a maximum. □

Assuming our upper bounds hold over the prime knots, Corollary 3.13 forces recursive sums whose base knot have a resultant unknot percentage less than or equal to 0.5 to similarly respect our proposed absolute bounds over the primes for the trefoil and figure-eight knots.

**Theorem 3.14.** *Let $K_1$ be a nontrivial prime knot with shape $S$. Then the resultant knot probability of $K_2$ from the recursive sum of $S$, $P(S^N \to K_2)$, has a limit* $\lim_{N\to\infty} P(S^N \to K_2) = 0$.

*Proof.* According to Theorem 3.12, after some number of steps, $x_N$ is monotone decreasing. As a probability, $x_N$ is also bounded below by zero; therefore the sequence has a limit $L$. Note the nontriviality of $K_1$ forces $0 < \alpha < 1$. Then

$$L = \lim_{N\to\infty} \alpha^N\beta + \lim_{N\to\infty} \alpha x_N = 0 + \alpha L.$$

This can only be true if $L = 0$. □

**Theorem 3.15.** *Let $K_1$ be a prime knot with shape $S$. Then for a nontrivial prime $K_2$ the resultant knot probability is*

$$P(S^N \to K_2) = N \cdot P(S \to U)^{N-1} P(S \to K_2).$$

*Proof.* We proceed by induction on $N$. Note $0\alpha^{-1}\beta = 0 = x_0$, and $1\alpha^0\beta = \beta = x_1$, showing agreement with our recursive formula for $P(S_1^N \to K_2)$ in the base cases.

Now suppose $x_k = k\alpha^{k-1}\beta$ for some $N = 0, 1, \ldots, k$. Then $x_{k+1} = \alpha^{k+1-1}\beta + \alpha x_k$. By the inductive hypothesis,

$$x_{k+1} = \alpha^k\beta + \alpha(k\alpha^{k-1}\beta) = \alpha^k\beta + k\alpha^k\beta = (k+1)\alpha^k\beta$$

and thus the formula holds for all $N$. □

**Theorem 3.16.** *Let $K_1$ be a nontrivial prime knot with shape $S$ with resultant unknot probability $\alpha$. Then the resultant knot probability for a nontrivial prime $K_2$*

*has a singular maximum at*

$$N = 1 \quad \text{if } \alpha < \tfrac{1}{2},$$
$$N = 2 \quad \text{if } \tfrac{1}{2} < \alpha < \tfrac{2}{3},$$
$$N = 3 \quad \text{if } \tfrac{2}{3} < \alpha < \tfrac{3}{4}.$$

*Proof.* If the resultant knot probability for $K_2$ has solely one maximum value after $N$ recursive sums, then the difference between the $N$-th and surrounding steps gives the inequalities

$$(N+1)\alpha^N\beta - N\alpha^{N-1}\beta < 0,$$
$$N\alpha^{N-1}\beta - (N-1)\alpha^{N-2}\beta > 0.$$

Solving each for $\alpha$ places bounds on what the resultant unknot probability can be while having a maximum solely at particular values of $N$:

$$\frac{N-1}{N} < \alpha < \frac{N}{N+1}.$$

Choosing low values of $N$ then produce the ranges above. Note if $\alpha$ is equivalent to one of these bounds, both $N$ and $N+1$ are (equivalent) maxima. □

Assuming Conjecture 1.4 holds, its corollary that the absolute maximum resultant unknot probability is 0.75 forces the following consequence of the conjecture.

**Consequence 3.17.** *All resultant knot probabilities of nontrivial prime knots from recursive sums have a maximum in the first four connected sums.*

*Proof.* As forced by Theorem 3.12, the resultant knot probability will attain a maximum at two adjacent steps if the difference between each is equal to zero ($N \geq 1$), or, specifically, if

$$(N+1)\alpha^N\beta - N\alpha^{N-1}\beta = 0 \quad \text{or} \quad (N+1)\alpha - N = 0.$$

Solving for $N$, the first step where the maximum occurs, as a function of $\alpha$, we get

$$N(\alpha) = \frac{\alpha}{1-\alpha}.$$

The derivative of $N(\alpha)$, $1/(1-\alpha)^2$, is positive for all $\alpha$, so the largest final step containing a maximum will occur for the largest possible $\alpha$, which we conjecture to be $\alpha = 0.75$. Since $N(0.75) = 3$, the final step containing a maximum will occur at $N = 4$, as the value of the resultant probability stays the same. □

Recursive sums (and other connect sums) have the potential to create counterexamples to our later proposed bounds on resultant trefoil and figure-eight probabilities. In those cases, we note that they apply to solely prime knots. These recursive sum counterexamples depend more on the unknot probability of the base knot shape

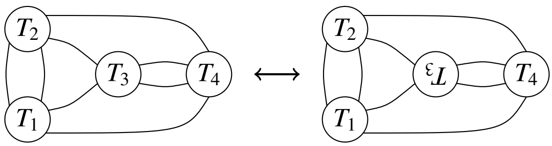



**Figure 8.** Two free knot diagrams with the same resultant knot probabilities.

than the base shape's resultant probability. For example, $7_1$ appears to produce the most trefoils at a rate of 32.8125%, but its recursive sum only produces a higher percentage (approximately 35.89%) for $(7_1)^2$. Meanwhile the recursive sum of the trefoil, with its absolute maximum resultant unknot probability of 75%, produces conjecture breaking results for $(3_1)^2$ through $(3_1)^6$.

We present two final general conjectures on resultant probabilities, based on observations of all free knot diagrams with up to nine crossings:

**Conjecture 3.18.** *For all free knot diagrams and nontrivial knots, the resultant knot probability is less than* 0.5.

There appear to be further manipulations we can perform within free knot diagrams which do not change their resultant knot probability. To describe these, we need the language of tangles.

**Definition 3.19.** A *tangle* is a portion of a knot or link contained in a circle which intersects the boundary at exactly four points. Reidemeister moves and planar isotopies are allowed in the interior of the circle.

A free tangle is a tangle with free crossings instead of true crossings. Observations show that 90° rotations and reflections across the diagonals of a tangle (i.e., an axis connecting two opposite anchor points) change the resultant knot probability. However, knot mutations [Viro 1976; Conway 1970], which are vertical and horizontal reflections as well as 180° rotations, seem not to change resultant knot probability.

**Conjecture 3.20.** *A knot and its mutants have the same resultant knot probability.*

For example, the free knot diagrams shown in Figure 8 with a variety of choices of $T_i$ (including nonsymmetric possibilities) were seen to have the same resultant knot probabilities.

## 4. Tangles and the $n_1$-knots

Recall that the notation for tangles introduced in [Conway 1970] gives us concise descriptors for a large family of knots. It is useful not only because of its brevity, but because it calculates an invariant of tangles, their continued fraction.

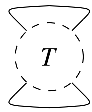


**Figure 9.** A tangle.

**Theorem 4.1** [Conway 1970]. *Two rational tangles are isotopic to each other if their continued fraction is equivalent.*

A tangle, shown in Figure 9, is made into a knot or link by taking its closure, which connects the upper and lower two pairs of exterior arcs. The continued fraction continues to differentiate between knots once we move to closures of tangles by the following process:

**Lemma 4.2** (from [Schubert 1956], as quoted in [Kauffman and Lambropoulou 2002]). *Suppose there exist two rational tangles with continued fractions $p/q$ and $p'/q'$, where $p$, $q$ and $p'$, $q'$ are relatively prime. Let $K(p/q)$ and $K(p'/q')$ be the knots formed by the closures of the respective rational tangles. Then $K(p/q)$ and $K(p'/q')$ are equivalent (up to isotopy) if and only if*

- $p = p'$,
- *either* $q \equiv q' \mod p$ *or* $qq' \equiv 1 \mod p$.

**Definition 4.3.** The $n_1$-knots are the closures of two-strand braids with $n$ crossings ($n$ is odd). These knots include the trefoil, pentafoil, $7_1$, $9_1$, and the unknot (with the first Reidemeister relation applied).

An odd number of crossings is required since an even number of crossings would require two components, making the result a link. The name $n_1$ comes from their position as the first entry in knot tabulations of knots with $n$ crossings.

**Theorem 4.4.** *The free $n_1$-knot produces $\binom{n}{(n-k)/2}$ left $k_1$-knots and $\binom{n}{(n-k)/2}$ right $k_1$-knots ($k \le n$).*

*Proof.* Previously, we have counted a knot and its reflection in the same category, even if it is chiral. Here, we count a knot and its reflection as distinct knots.

A braid can have two types of crossings, "positive slope" and "negative slope", referring to the slope of the overstrand. Given a free $n_1$-knot, we may choose $\ell$ of its crossings to be positive slope crossings and the remaining $n - \ell$ to be negative slope crossings. Then, if $0 < \ell < n$, we may find a positive crossing next to a negative crossing, and remove the pair via Reidemeister II. Proceeding thus until all crossings of one type have been removed, we are left with either $n - 2\ell$ or $2\ell - n$ crossings of a single type. Substituting $\ell = (n-k)/2$ or $\ell = (n+k)/2$ produces the formula stated above. □

**Corollary 4.5.** *The expected number of crossings for a resultant of an $n_1$-knot is* $2^{1-n}\sum_{k=3}^{n} k\binom{n}{(n-k)/2}$, *where the values of $k$ are odd.*

*Proof.* Since free $n_1$-knots produce an equal number of right and left $k_1$-knots, the probability of getting a resultant with $k$ crossings is $\frac{2}{2^n}\binom{n}{(n-k)/2}$. The free 2-braid knot diagram with one crossing is the unknot; thus its probability has a coefficient of zero in the expectation value. Then the expectation value sum, over the allowed odd crossing values, starts at the trefoil and goes up to the $n_1$-knot. □

**Theorem 4.6.** *For fixed odd $k$, the limit as the number of crossings $n$ goes to infinity of the resultant $k_1$-knot probability is $0$ for free $n_1$-knots.*

*Proof.* Applying Stirling's approximation to the resultant $k_1$-knot probability leads to

$$\frac{2}{2^n}\binom{n}{(n-k)/2} \sim \sqrt{\frac{8}{\pi}}\,\frac{n-k}{(n+k)^2}\,\frac{n^{n+1/2}}{(n-k)^{n/2}(n+k)^{n/2}}.$$

The highest power of $n$ occurring in the denominator is $n+2$, and is larger than the highest power of $n$ in the numerator $\big($which is $n+\frac{3}{2}\big)$. Thus the resultant $k_1$-knot probability goes to zero as $n$ goes to infinity. □

**Conjecture 4.7.** *Among all minimal prime free knot diagrams with $n$ and $n+1$ crossings, where $n$ is odd and $n \geq 3$, the free $n_1$-knot has the most trefoil descendants.*

To create a trefoil, an assignment of crossings must (after Reidemeister II cancellations) result in three alternating crossings in a row. When trying to find assignments that create such a shape, the $n_1$-knots should then have an advantage.

Conjecture 4.7 would also allows us to strengthen Theorem 4.6's results by applying it to all knots, forcing the limit of the resultant $k_1$-knot probability as the number of crossings $n$ goes to infinity to be 0.

**Consequence 4.8.** *For prime knots, the absolute maximum $k_1$-knot probability is* $2^{1-k^2}\binom{k^2}{(k^2-k)/2}$, *and consequently, the maximum trefoil percentage is* 32.8125.

*Proof.* We create a sequence $g_n$ calculating the resultant $k_1$-knot probability for $n_1$-knots, where $n$ must be incremented by 2:

$$g_n = \frac{2\binom{n}{(n-k)/2}}{2^n} = \frac{1}{2^{n-1}}\binom{n}{(n-k)/2}.$$

The next term in the sequence can be written as

$$g_{n+2} = \frac{(n+2)(n+1)}{(n+2-k)(n+2+k)}\,g_n.$$

This sequence will be monotonically decreasing when $n > k^2$ and monotonically increasing when $n < k^2-2$. Observing that $g_{k^2-2} = g_{k^2}$, we have two occurrences

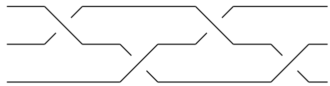


**Figure 10.** The minimal braid representation of the figure-eight knot.

of the maximum probability for a given $k_1$-knot:

$$2^{3-k^2}\binom{k^2-2}{(k^2-k-2)/2} = 2^{1-k^2}\binom{k^2}{(k^2-k)/2}.$$

Then for the trefoil ($k=3$), $7_1$ and $9_1$ have the maximum trefoil percentage of 32.8125. □

**Observation 4.9.** *Let K be a minimal knot with four or more crossings, which is neither an $n_1$-knot nor a connect sum of $n_1$-knots. Then K has some assignment of crossings making the figure-eight knot.*

The above statement is called an observation, not a conjecture, because it is proven in [Cantarella et al. 2017] using results from [Taniyama 1989]. However, it still seems desirable to us to have a more direct proof of this, perhaps along the lines of the proof of Theorem 3.5 above.

Theorem 4.4 confirms that all resultants of free $n_1$-knots are also 2-braid knot diagrams, so no figure-eights are produced. These appear to be the only free knots without a figure-eight resultant. This is sensible since they are the only knots with braid index of 2.

The figure-eight knot has the minimal braid representation shown in Figure 10. If one of the outer strands of a minimal braid diagram was connected to its neighboring internal strand by only one crossing, the external strand could be removed via braid stabilization, so a braid index greater than 3 forces the existence of more than four free crossings. Then seemingly there should be a route to unknotting some remaining crossings to produce a figure-eight resultant.

Based on our experimental evidence, we propose slightly stronger upper bounds on the unknot probability as this would require four nontrivial resultants of a given free knot diagram.

**Conjecture 4.10.** *The resultant figure-eight probability is less than or equal to the resultant trefoil probability for a free knot diagram coming from algebraic tangles.*

The hypothesis concerning algebraic tangles is included because we know of exactly one counterexample to this conjecture: the knot $9_{40}$, which is not the closure of an algebraic tangle. It has 66 resultant trefoils and 78 resultant figure-eights. Yet this conjecture holds for all other free knots with nine or fewer crossings.

**Consequence 4.11.** *For all algebraic free knot diagrams with $n$ and $n+1$ crossings, where $n$ is odd and $n \geq 3$, the trefoil probability of the $n_1$-knot is the upper bound on figure-eight probability.*

*Proof.* This extends the previous results from trefoils to figure-eights using the above conjecture over all algebraic diagrams. □

The largest resultant figure-eight percentage for all knots through eight crossings is shared by $7_7$ and $8_{12}$ at 15.625%, so there is likely a more restrictive bound to be found.

**Conjecture 4.12.** *The absolute maximum resultant figure-eight percentage is 15.625%.*

When moving beyond the algebraic knots, this number appears to remain the upper bound, as $9_{40}$ and $9_{47}$ have respective resultant figure-eight percentages of 15.234% and 13.281%.

## 5. The $(k, n)$-knots

**Definition 5.1.** The $(2, n)$-knots (Figure 11, left), closures of the $(2, n)$-tangles, include the trefoil, the figure-eight, $5_2$, $6_1$, $7_2$, $8_1$, $9_2$, and so on.

One may think about generating these knots by repeatedly twisting a loop, and then coupling the two ends together. This construction method demonstrates that these knots should produce a lot of unknots! Any time the upper two strands can be separated, every assignment of the remaining crossings must give an unknot.

**Definition 5.2.** The $(k, n)$-knots (Figure 11, right) are the generalization of the $(2, n)$-knots, and include knots like $7_3$ and $8_3$. One of $k$ or $n$ must be even or else we get a link, not a knot, from the closure.

Using continued fractions, Lemma 4.2 shows $(2, n)$-knots are nontrivial if $n \geq 1$, and distinct for distinct $n$. It also tells us the $(k, n)$-knots are distinct for $k \geq 3$, $n \geq 3$.

**Theorem 5.3.** *The resultant knot probability of a free $(k, n)$-knot is the same as the resultant knot probability for a free $(n, k)$-knot.*

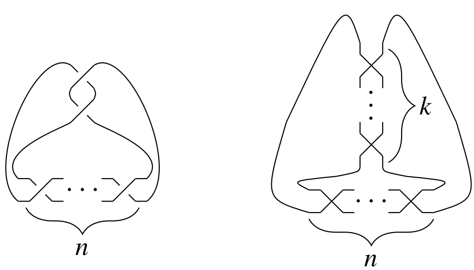



**Figure 11.** A $(2, n)$-knot (left) and a $(k, n)$-knot (right).

<table>
<tr><td colspan="2">unknot</td><td>(2, $k$)-knot</td><td>(2, ($k-1$))-knot</td></tr>
<tr><td>$n$ even</td><td>$n$ odd</td><td colspan="2">$n, k$ same parity</td></tr>
<tr><td>$2^{n+1}+2\binom{n}{n/2}$</td><td>$2^{n+1}+2\binom{n}{(n-1)/2}$</td><td colspan="2">$2\binom{n}{(n-k)/2}$</td></tr>
</table>

**Table 1.** The complete resultants of the free $(2, n)$-knot, $k \leq n$.

*Proof.* Begin with a free $(k, n)$-knot as depicted in Figure 11. Via a spherical isotopy, bring the far right strand around the rest of the free knot diagram so it is now to the left of everything else. Next isotope the resulting diagram so the $k$-tangle is made horizontal by a 90° clockwise rotation, and the $n$-tangle is made vertical by a 270° clockwise rotation. The result is the $(n, k)$-free knot diagram. □

In this and the following section, we present many theorems without proofs, when those proofs are straightforward computations, most of which rely chiefly on the results of the explication of the $n_1$-knot within each free knot structure.

**Theorem 5.4.** *The free $(2, n)$-knot's complete resultants $(k \leq n)$ are given in Table 1. (To calculate the resultant probability for an even/odd resultant with $\ell$ crossings when both n and k are odd/even, let $k = \ell + 1$.)*

**Theorem 5.5.** *The number of unknots produced by a $(k, n)$-free knot diagram with $k, n$ both even is*

$$2^k\binom{n}{n/2}+2^n\binom{k}{k/2}-\binom{n}{n/2}\binom{k}{k/2}$$

*and with k odd and n even is*

$$2^k\binom{n}{n/2}+2\binom{k}{(k-1)/2}\binom{n}{(n-2)/2}.$$

**Corollary 5.6.** *The expected number of crossings for a resultant of a $(2, n)$-knot is*

$$2^{-(n+1)}\sum_{k=2}^{n}(2k+3)\binom{n}{(n-k)/2},$$

*where the values of k and n are even, and*

$$2^{-(n+1)}\left(3\binom{n}{(n-1)/2}+\sum_{k=3}^{n}(2k+3)\binom{n}{(n-k)/2}\right),$$

*where the values of k and n are odd.*

**Theorem 5.7.** *The limit of resultant unknot probability as the number of crossings n $(k \leq n)$ goes to infinity is 0.5 for $(2, n)$-knots.*

*Proof.* The resultant unknot probabilities for a $(2, n)$-knot are

$$\frac{1}{2}+\frac{1}{2^{n+1}}\binom{n}{n/2}, \quad \frac{1}{2}+\frac{1}{2^{n+1}}\binom{n}{(n-1)/2}$$

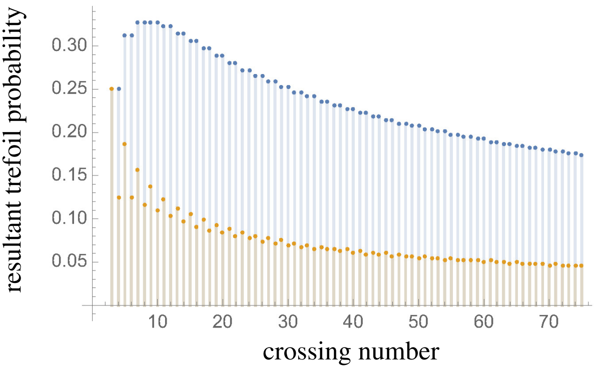



**Figure 12.** Resultant trefoil probabilities for the $n_1$-knots (blue) and $(2, n)$-knots (orange).

when $n$ is even and odd, respectively. Applying the asymptotic behavior in Theorem 4.6 for $k = 0$ and $k = 1$ to the second terms, only the $\frac{1}{2}$ term remains for both odd and even $n$. □

**Conjecture 5.8.** *The free $(2, n)$-knots realize the upper bound on the unknots for minimal free knot diagrams with $n+2$ crossings.*

**Consequence 5.9.** *The absolute maximum value for the unknot percentage from a nontrivial knot is 75%.*

*Proof.* Similar to Consequence 4.8, we define infinite series $e_n$ and $o_n$ equivalent to the probability of the unknot for the $(2, n)$-knots for even and odd $n$:

$$e_n = \frac{1}{2} + \frac{1}{2^{n+1}}\binom{n}{n/2}, \quad o_n = \frac{1}{2} + \frac{1}{2^{n+1}}\binom{n}{(n-1)/2}.$$

Both series are easily seen to be monotonically decreasing, so the $(2, 1)$-knot — i.e., the trefoil — has the maximum probability, at 75%. □

**Conjecture 5.10.** *The free $(2, n)$-knots are the lower bound for trefoils for minimal prime algebraic free knot diagrams with $n+2$ crossings.*

As so many assignments of the free $(2, n)$-knot are forced to be the unknot, this does not leave much room for trefoils. If true, the free $(2, n)$-knots are not the only knots that realize the lower bound. For example, $7_2$ shares the same resultant trefoil percentage, 15.625%, with $7_4$ and $7_7$. Again, we insert the algebraic knot qualifier for the lone counterexample of $9_{40}$.

Between the two trefoil bounding conjectures, the space of possible trefoil probabilities would be known. Figure 12 shows these possible ranges of probabilities for free knots with up to 75 crossings in between those for the $n_1$-knots (blue) and $(2, n)$-knots (orange).

## 6. The (2, 1, $n$)-knots

**Definition 6.1.** The $(2, 1, n)$-knots are again the closure of the $(2, 1, n)$-tangle when $n$ is odd; see Figure 13. Examples include the figure-eight, $6_2$, and $8_2$.

In applying this section's formulas, if a term's lower integer index for its binomial coefficient (the $k$ in the "$n$ choose $k$" language) is negative, that term should be understood to be ignored.

**Theorem 6.2.** *A free* $(2, 1, n)$*-knot produces* $12\binom{n}{(n-1)/2} + 2\binom{n}{(n-3)/2}$ *unknots.*

*Proof.* We break into eight cases, depending on the assignment of the crossings in the $(2, 1)$-portion of the tangle, and then the computation is very similar to the computations above. □

**Theorem 6.3.** *A free* $(2, 1, n)$*-knot produces*

$$8\binom{n}{(n-3)/2} + 2\binom{n}{(n-1)/2} + 2\binom{n}{(n-5)/2}$$

*trefoils, and, in general,*

$$8\binom{n}{(n-k)/2} + 2\binom{n}{(n-k+2)/2} + 2\binom{n}{(n-k-2)/2}$$

$k_1$*-knots* ($k \le n+2$).

*Proof.* First consider the different possible assignments of the $(2, 1)$-tangles shown in Figure 14.

The first four assignments (red) of the free $(2, 1, n)$-tangle produce only closures of two-strand braids, so we get $4 \cdot 2\binom{n}{(n-3)/2}$ trefoils. In the green assignments, a $\pm 2$-tangle can be added to a $\mp 5$-tangle or a $\pm 1$-tangle to produce a trefoil: there are $2\binom{n}{(n-5)/2}$ ways to make the free $n$-tangle into a $\mp 5$-tangle, and $2\binom{n}{(n-1)/2}$ ways to make the free $n$-tangle into a $\pm 1$-tangle.

If one's goal is to get a $k$-tangle, there are $4 \cdot 2\binom{n}{(n-k)/2}$ ways to get this from the red assignments, and $2\binom{n}{(n-k-2)/2} + 2\binom{n}{(n-k+2)/2}$ ways from the green.

The final two assignments (blue) are the closures of the $(\pm 2, \pm 1, m)$-tangles, which have continued fractions

$$m + \frac{1}{\pm 1 \pm \frac{1}{2}} = m \pm \tfrac{2}{3} = \frac{3m \pm 2}{3}.$$

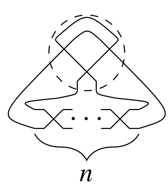



**Figure 13.** A $(2, 1, n)$-knot.

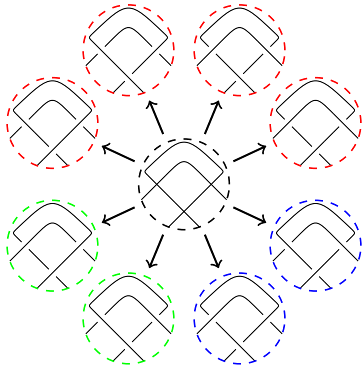


**Figure 14.** Possible arrangements of (2, 1)-tangles.

We appeal to Lemma 4.2, as any $k_1$-knot would be $K(k/1)$, where $k$ is an odd integer, so $q = 1$ and $q' = 3$. Thus these assignments cannot produce any $k_1$-knots, as the difference of 3 and 1 is even, so the odd $k$ ensures $1 \not\equiv 3 \bmod k$ and $3 \not\equiv 1 \pmod{k}$. $\square$

**Theorem 6.4.** *A free* $(2, 1, n)$*-knot produces* $2\binom{n}{(n-1)/2}$ *figure-eights, and* $2\binom{n}{(n-k)/2}$ *of* $(2, 1, k)$*-knots,* $k \leq n$.

*Proof.* As in the proof above, we focus on the resolution of the (2, 1)-tangle. The red and green assignments are all $k_1$-knots, so any figure-eight knot must come from a blue assignment. This is then a straightforward combinatorics exercise. $\square$

**Theorem 6.5.** *A free* $(2, 1, n)$*-knot produces* $2\binom{n}{(n-k)/2}$ $(3, k-1)$*-knots for* $3 \leq k \leq n$, *$k$ odd.*

*Proof.* The only remaining unexamined combination of this knot's component tangles are when $(\pm 2, \pm 1)$-tangles are connected to $\mp k$-tangles $(3 \leq k \leq n)$. Without loss of generality, we'll examine the $(-2, -1, k)$-case with continued fraction $\frac{3k-2}{3}$. Now using Lemma 4.2 in a positive sense, we can identify the closures of this tangle as isotopic to the closures of $(3, \ell)$-tangles $\bigl($whose continued fraction is $\ell + \frac{1}{3} = \frac{3\ell+1}{3}\bigr)$ since their numerators are equal when $\ell = k - 1$, and their identical denominators trivially satisfy the second condition by reflexivity. Since the (2, 1)-tangles are already assigned, the number of each resultant formed depends on the number of ways to assign the $n$-tangle to be $k$ crossings long, namely $2\binom{n}{(n-k)/2}$ ways. $\square$

This completes the elucidation of the $(2, 1, n)$-knot's resultants.

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