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Simple graphs of order 12 and minimum degree 6 contain K_6 minors

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We prove that every simple graph of order 12 which has minimum degree 6 contains a K_6 minor, thus proving Jørgensen's conjecture for graphs of order 12. In the process, we establish several lemmata linking the existence of K_6 minors for graphs to their size or degree sequence, by means of their clique sum structure. We also establish an upper bound for the order of graphs where the 6-connected condition is necessary for Jørgensen's conjecture.

1. Introduction

All the graphs considered in this article are simple (nonoriented, without loops or multiple edges). For a graph G, *a minor of* G is any graph that can be obtained from G by a sequence of vertex deletions, edge deletions, and simple edge contractions. A simple edge contraction means identifying its endpoints, deleting that edge, and deleting any double edges thus created. A graph G is called *apex* if it has a vertex v such that $G - v$ is planar, where $G - v$ is the subgraph of G obtained by deleting vertex v and all edges of G incident to v. Jørgensen [1994] stated the following conjecture:

Conjecture 1. Let G be 6-connected graph which does not have a K_6 minor. Then G is apex.

This result relates to Hadwiger's conjecture [1943], which states:

Conjecture 2. For every integer $t \geq 1$, if a loopless graph G has no K_t minor, then it is $(t-1)$-colorable.

Conjecture 2 is known to be true for $t \leq 6$. For $t = 5$, the conjecture is equivalent to Appel and Haken's 4-Color theorem [1989]. For $t = 6$, Robertson, Seymour, and

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Thomas [1993] proved it using a result of Mader. Mader [1968b] proved that a minimal counterexample to Conjecture 2 for $t = 6$ has to be 6-connected. Together with Jørgensen's conjecture, it would provide another proof that Conjecture 2 holds for $t = 6$, along with more information about the structure of graphs with no K_6 minors.

Jørgensen himself took steps towards proving Conjecture 1. In [Jørgensen 1994], he proved that every graph G with at most 11 vertices and minimal degree $\delta(G)$ at least 6 is contractible to a K_6. In his proof, he used the following result of [Mader 1968a]:

Theorem 3. *Every simple graph with minimal degree at least* 5 *either has a minor isomorphic to* K_6^- *or it has a minor isomorphic to the icosahedral graph.*

The icosahedral graph is the only 5-regular planar graph on 12 vertices. Mader [1968a] also proved the following theorem.

Theorem 4. *For every integer* $2 \le t \le 7$ *and every simple graph* G *of order* $n \ge t-1$ *which has no minor isomorphic to* K_t, G *has at most* $(t-2)n - \binom{t-1}{2}$ *edges.*

Note that for $t = 6$, the theorem implies that every graph G of order n and size $4n - 9$ or more has a K_6 minor.

Jørgensen [1988] classified the graphs of order n and size $4n - 10$.

Theorem 5. *Let* p *be a natural number*, $5 \le p \le 7$. *Let* G *be a graph with* n *vertices and* $(p-2)n - \binom{p}{2}$ *edges that is not contractible to* K_p. *Then either* G *is an* MP_{p-5}*-cockade or* $p = 7$ *and* G *is the complete* 4*-partite graph* $K_{2,2,2,3}$.

For $p = 6$, this theorem shows that any graph G of order n and size $4n - 10$ either contains a K_6 minor, or it is an MP_1-cockade. The following is Jørgensen's definition of an MP_1-cockade.

Definition 6. MP_1-cockades are defined recursively as follows:

(1) K_5 is an MP_1-cockade and if H is a 4-connected maximal planar graph then $H * K_1$ is an MP_1-cockade.

(2) Let G_1 and G_2 be disjoint MP_1-cockades, and let x_1, x_2, x_3, and x_4 be the vertices of a K_4 subgraph of G_1 and let y_1, y_2, y_3, and y_4 be the vertices of a K_4 subgraph of G_2. Then the graph obtained from $G_1 \cup G_2$ by identifying x_j and y_j, for $j = 1, 2, 3, 4$, is an MP_1-cockade.

For two graphs G_1 and G_2, we denote by $G_1 * G_2$ the graph with vertex set $V(G_1) \sqcup V(G_2)$ and edge set $E(G_1) \sqcup E(G_2) \sqcup E'$, where E' is the set of edges with one endpoint in $V(G_1)$ and the other endpoint in $V(G_2)$. In $G_1 * v$, we call v *a cone* over G_1. A graph G is the *clique sum* of G_1 and G_2 over K_p if $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$ and the subgraphs induced by $V(G_1) \cap V(G_2)$ in both G_1 and G_2 are complete of order p. In this context, an MP_1-cockade is either a cone over a 4-connected maximal planar graph or the clique sum over K_4 of two smaller MP_1-cockades.

Kawarabayashi, Norine, Thomas, and Wollan [2018] proved that Conjecture 1 holds for sufficiently large graphs. Little is known about the validity of Conjecture 1 for small order graphs. In this paper, we prove that Jørgensen's conjecture holds for graphs of order 12 in a more general setting.

Theorem 7. *Let G be a simple graph of order 12 and assume that $\delta(G) \geq 6$, where $\delta(G)$ denotes the minimal degree of G. Then G contains a K_6 minor.*

Note that the theorem implies Jørgensen's conjecture is vacuously true for graphs of order 12.

2. Main theorem

For a graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. The size of $V(G)$ is called *the order* of G, and the cardinality of $E(G)$ is called *the size* of G. For $n \geq 1$, K_n denotes the complete graph of order n and K_n^- denotes the complete graph of order n with one edge removed. If $v_1, v_2, \dots, v_k$ are vertices of G, then $\langle v_1, v_2, \dots, v_k \rangle_G$ denotes the subgraph of G induced by these vertices. If v is a vertex of G, then $N_G[v]$ is the subgraph of G induced by v and the vertices adjacent to v in G (the closed neighborhood of v). Let $N_G(v)$ denote the subgraph of G induced by all the vertices adjacent to v (the open neighborhood of v). If S is a subset of $V(G)$, then $G - S$ is the subgraph of G obtained by deleting all of the vertices in S and all the edges of G to which S is incident.

The following lemma is a corollary of Theorem 4.

Lemma 8. *Let G be a simple graph of order n and size $4n - 10$. If $G - v$ is planar, then v cones over $G - v$.*

Proof. Since $G - v$ is planar of order $n - 1$, it has at most $3(n-1) - 6 = 3n - 9$ edges. This implies that v has at least $4n - 10 - (3n - 9) = n - 1$ neighbors, and the conclusion follows. □

Proof of Theorem 7. Let G denote a simple graph of order 12 and minimal degree $\delta(G)$ at least six. It follows that G has at least size 36. By Theorem 4, if the size of G is at least 39, then G contains a K_6 minor. We shall prove Theorem 7 by considering the size of G, $36 \leq |E(G)| \leq 38$.

Case 1: Assume $|E(G)| = 38$. By Theorem 5, either G contains a K_6 minor, or G is apex, or G is the clique sum over K_4 of two MP_1 cockades.

If G is isomorphic to $H * K_1$, where H is a maximal planar graph on 11 vertices, then $\delta(H) \geq 5$ and, by Theorem 3, it follows that H has a K_6^- minor and thus G has a K_6 minor.

Assume that G is the clique sum over $S \simeq K_4$ of two MP_1-cockades. If $G - S$ has more than two connected components, then at least one of them has at most two vertices. But this contradicts the fact that $\delta(G) \geq 6$. So $G - S = Q_1 \sqcup Q_2$. Furthermore, unless $|Q_1| = |Q_2| = 4$, the graph either contains a K_7 subgraph (if $|Q_1|$=3), or $\delta(G) < 6$ (if $1 \leq |Q_1| \leq 2$). Since $\delta(G) \geq 6$, it follows that each vertex of Q_i connects to at least two other vertices of Q_i, for $i = 1, 2$ respectively.

Without loss of generality, let $Q_1 = \langle v_1, v_2, v_3, v_4 \rangle_G$. If Q_i is not isomorphic to K_4 for any of $i = 1, 2$, say $v_1 v_2 \notin E(Q_1)$, since v_1 and v_2 must both connect to both v_3 and v_4, contracting the edges $v_1 v_3$ and $v_2 v_4$ produces a minor of G which contains a K_6 subgraph induced by v_1, v_2, and the four vertices of S. If, on the other hand, $Q_1 \simeq Q_2 \simeq K_4$, as $\delta(G) \geq 6$, it follows that there are at least 12 edges between each of the Q_i and S. That would imply that $|E(G)| \geq 6 + 12 + 6 + 12 + 6 = 42$, a contradiction. It follows that for $|E(G)| = 38$, G has a K_6 minor.

Case 2: Assume $|E(G)| = 37$. Since $\delta(G) \geq 6$, it follows that the degree sequence of G is either (6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 8) or (6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 7, 7). In either of the situations, we shall need the following lemma:

Lemma 9. *Let M denote a graph of order* 11 *and size* 34 *such that* $\delta(M) \geq 5$. *Assume that M is not apex and has at most four vertices of degree* 5. *Then M contains a K_6 minor.*

Proof. By Theorem 5, either M contains a K_6 minor or is an MP_1-cockade. Since M is not apex, it follows that M is the clique sum over $S \simeq K_4$ of two MP_1-cockades. If $M - S$ has more than two connected components, $Q_1, Q_2, \ldots$, then at least one of them, say Q_1, has at most two vertices. As $|Q_1| = 1$ would violate the condition $\delta(M) \geq 5$, it follows that $|Q_1| = 2$ and the subgraph of M induced by Q_1 and S forms a K_6. So $M - S = Q_1 \sqcup Q_2$ and, without loss of generality, $Q_1 = \langle v_1, v_2, v_3 \rangle_M$. Unless $Q_1 \simeq K_3$, since $\delta(M) \geq 5$, it follows that at least two vertices of Q_1 connect to all the vertices of S and thus, via an edge contraction, they induce a K_6 minor of M.

If $Q_1 \simeq K_3$, there have to be exactly nine edges connecting the vertices of Q_1 to those of S. If there are more than nine, the subgraph induced by the vertices of Q_1 and S has seven vertices and more than $3 + 9 + 6 = 18$ edges; thus it contains a K_6 minor by Theorem 4. If there are less than nine, then at least one of the vertices of Q_1 has degree less than 5. So all the vertices of Q_1 have degree 5, and the subgraph induced by the vertices of Q_1 and S has exactly 18 edges. If L denotes the set of edges connecting the vertices of Q_2 to the vertices of S, then $|L| + |E(Q_2)| = 34 - 18 = 16$. On the other hand, since Q_2 can have at most one vertex of degree 5 in M, it follows that $|L| + 2|E(Q_2)| \geq 6 + 6 + 6 + 5 = 23$. Subtracting the last two equalities we get $|E(Q_2)| \geq 7$, a contradiction as Q_2 has four vertices. □

Assume the vertex degree sequence for G is (6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 8). Furthermore, without loss of generality, we may assume $\deg_G(v_1) = 6$, $\deg_G(v_8) = 8$, and that $N_G[v_1] = \langle v_1, \ldots, v_7 \rangle_G$. Let $N = \langle v_2, v_3, \ldots, v_7 \rangle_G$, $H = \langle v_8, \ldots, v_{12} \rangle_G$, and let L denote the set of edges of G with one endpoint in N and the other in H. The handshaking lemma provides the following relations between the sizes of $E(N)$, L, and $E(H)$:

$$2|E(N)| + |L| = 30, \quad 2|E(H)| + |L| = 32.$$

If $|E(H)| = 10$, that is, $H \simeq K_5$, as every vertex of H must have at least two neighbors in N ($\delta(G) \geq 6$), contracting the edges of $N_G[v_1]$, produces a K_6 minor of G.

If $|E(H)| \leq 9$, then $|L| \geq 14$ and thus $|E(N)| \leq 8$. It follows that there is a vertex of N, say v_2, such that $\deg_N(v_2) \leq 2$. If $\deg_N(v_2) < 2$, contracting the edge $v_1 v_2$ would produce a minor of G of order 11 and size at least 35, which would contain a K_6 minor by Theorem 4.

If $\deg_N(v_2) = 2$, then contracting the edge $v_1 v_2$ would produce a minor M of G of order 11 and size precisely 34. Furthermore, since v_2 neighbors exactly three vertices of H, the maximum degree of M is 8, so it cannot be apex, according to Lemma 8. Lastly, M has at most two vertices of degree 5, since $\deg_N(v_2) = 2$. By Lemma 9, M has a K_6 minor, and therefore so does G.

Assume the vertex degree sequence for G is (6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 7, 7). If the degree-7 vertices are connected in G, deleting the edge connecting them would produce a 6-regular subgraph of order 36, to be dealt with in the last case of the proof. So, without loss of generality, assume that $\deg_G(v_1) = 6$, $\deg_G(v_8) = 7$, and $N_G[v_1] = \langle v_1, \ldots, v_7 \rangle_G$. Let $N = \langle v_2, v_3, \ldots, v_7 \rangle_G$, $H = \langle v_8, \ldots, v_{12} \rangle_G$, and let L denote the set of edges of G with one endpoint in N and the other in H. If $\deg(v_i) = 7$, for some $9 \leq i \leq 12$, the same argument as before shows that $|E(N)| \leq 8$ and thus G contains a K_6 minor. So we may assume $\deg_G(v_7) = 7$ and $v_7 v_8 \notin E(G)$. Using the handshaking lemma, we get

$$2|E(N)| + |L| = 31, \quad 2|E(H)| + |L| = 31.$$

If $|E(H)| = 10$, contracting the edges of $N_G[v_1]$, produces a K_6 minor of G.

If $|E(H)| \leq 9$, then $|L| \geq 13$ and thus $|E(N)| \leq 9$. If $|E(N)| \leq 8$, it follows that there is a vertex of N, v_i, such that $\deg_N(v_i) \leq 2$. If $\deg_N(v_i) < 2$, contracting the edge $v_1 v_i$ would produce a minor of G of order 11 and size at least 35, which would contain a K_6 minor by Theorem 4.

Assume $\deg_N(v_i) = 2$. Contracting the edge $v_1 v_i$ produces a minor M of G of order 11 and size 34. Moreover, since for $2 \leq j \leq 7$, v_j neighbors at most four of the vertices of H, M cannot be apex. Lastly, M has at most two vertices of degree 5, since $\deg_N(v_i) = 2$. By Lemma 9, M has a K_6 minor, and therefore so does G.

It follows that N is 3-regular, $L = 13$ and $|E(H)| = 9$; that is, $H \simeq K_5^-$. If the missing edge of H has v_8 as its endpoint, and since $\deg_G(v_8) = 7$, it follows that v_8 neighbors four vertices of N. As the other endpoint, say v_9, neighbors three vertices

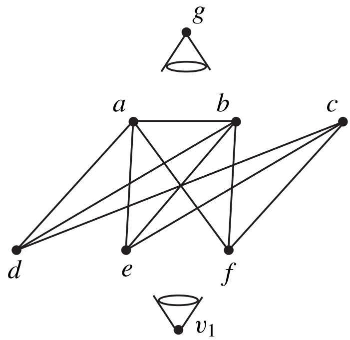

Figure 1. Contracting the edges cd and eg produces a K_6-minor.

of N, it follows that there exists $2 \le i \le 6$ such that v_i is a common neighbor of v_8 and v_9. Contracting the edges $v_i v_8$ and $v_1 v_j$, for $2 \le j \le 7$, $j \ne i$, one obtains a K_6 minor of G, as every vertex of H neighbors at least one of the v_j.

Assume the missing edge of H is $v_9 v_{10}$. Since N is 3-regular of order 6, it is isomorphic to either $K_{3,3}$ or the prism graph. If $N \simeq K_{3,3}$, as v_8 neighbors three vertices of N, contracting the edge connecting v_8 to one of its neighbors in N and all the edges in the subgraph induced by v_9, v_{10}, v_{11}, and v_{12}, produces a minor of G isomorphic to the graph in Figure 1. This minor has a K_6-minor.

If N is isomorphic to the prism graph in Figure 2, with the same labeling, for any vertex of v of H, the subgraph of G induced by its neighbors among the vertices of N must be complete (clique), since otherwise contracting v to one of its neighbors and the edges of $H - v$ produces a minor of G isomorphic to the graph in Figure 3. This graph has a K_6-minor.

If v_9 and v_{10} share a common neighbor among the vertices of N, say v_i, then contracting the edge $v_i v_9$ and $v_1 v_j$, for $2 \le j \le 7$, $j \ne i$, produces a K_6-minor. If v_9 and v_{10} have no common neighbor among the vertices of N, since v_9, v_{10} and v_8 each have exactly three neighbors among the vertices of N, and v_7 is not adjacent to v_8, up to a relabeling of v_9 and v_{10}, it must be that v_8 and v_9 together with v_2, v_4, and v_6 induce a K_5 subgraph of G. Contracting all the edges of $\langle v_1, v_7, v_{10}, v_{11}, v_{12} \rangle_G$ produces a K_6-minor of G.

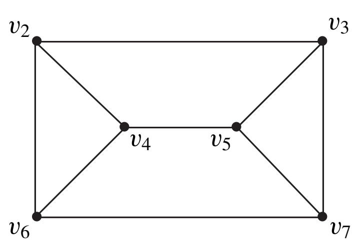

Figure 2. The graph N, the open neighborhood of v_1.

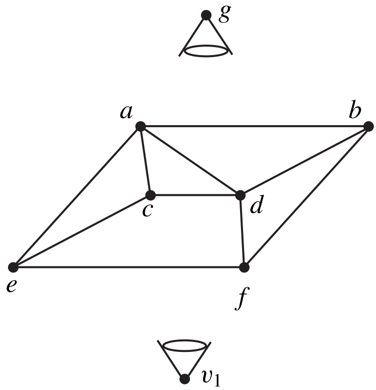

Figure 3. Contracting the edges bg and ef produces a K_6-minor.

<u>Case 3</u>: Assume $|G| = 36$; that is, G is a 6-regular graph. Let $v_2, v_3, \ldots, v_7$ be the neighbors of some vertex v_1 in G and let $N = \langle v_2, v_3, \ldots, v_7 \rangle_G = N_G(v_1)$ be the open neighborhood of v_1. Let $H = \langle v_8, \ldots, v_{12} \rangle_G$ and let L denote the subset of $E(G)$ of edges having one endpoint in N and the other in H. Then, as before, since the degree of every vertex in G is 6, we have

$$2|E(H)| + |L| = 30, \quad 2|E(N)| + |L| = 30;$$

thus $|E(N)| = |E(H)|$. If $|E(H)| = 10$, then $H \simeq K_5$. Since $\delta(G) = 6$ by hypothesis, each vertex in H must be adjacent to a vertex in N. Contracting the edges of $N_G[v_1]$ produces a K_6 minor of G. It follows that $|E(N)| \leq 9$ and, unless N is 3-regular, there exists at least a vertex of N which has at most two neighbors in N. There are two possible remaining cases: either the open neighborhood of every vertex of G is 3-regular, or there is a vertex of G whose open neighborhood contains a vertex of degree at most 2. Jørgensen [1994] proved that in a 6-regular graph, if the open neighborhood of every vertex of G is 3-regular, then any connected component of the graph is isomorphic to either $K_{3,3,3}$ or the complement of the Petersen graph. Since both contain K_6 minors, it suffices to consider the case $\deg_N(v_i) \leq 2$, for some $2 \leq i \leq 7$.

If, for some $2 \leq i \leq 7$, $\deg_N(v_i) = 0$, then contracting the edge $v_1 v_i$ produces a minor of G of order 11 and size 35. By Theorem 4, this minor has a K_6 minor.

If, for some $2 \leq i \leq 7$, $\deg_N(v_i) = 1$, then contracting the edge $v_1 v_i$ produces a minor M of G of order 11 and size 34. Furthermore, the degree sequence of this minor would be (5, 6, 6, 6, 6, 6, 6, 6, 6, 6, 9); hence, by Lemma 8, M cannot be apex. By Lemma 9, M would have a K_6 minor.

We may then assume that, for $2 \leq i \leq 7$, $\deg_N(v_i) \geq 2$ and, without loss of generality, the neighbors of v_2 in N are v_3 and v_4.

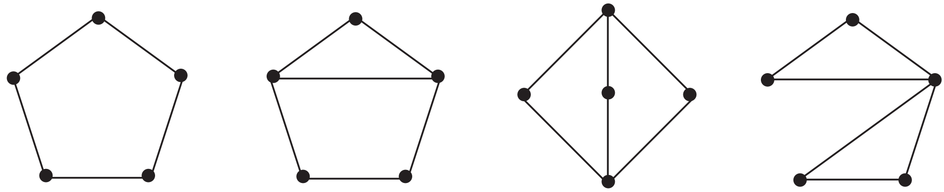

Figure 4. Graphs of order 5 and size at most 6, with minimum degree 2.

Subcase 3.1: Assume that $v_3v_4 \in E(G)$. Contracting the edge v_1v_2 produces a minor M of G of order 11 and size 33. Furthermore, the degree sequence of this minor is $(5, 5, 6, 6, 6, 6, 6, 6, 6, 6, 8)$. Via a relabeling, for the rest of this subcase, we may assume that $\deg_M(v_1) = \deg_M(v_2) = 5$, and that $\deg_M(v_6) = 8$. As before, let $N = \langle v_2, v_3, v_4, v_5, v_6\rangle_M$ be the subgraph of M induced by all the neighbors of v_1, $H = \langle v_7, \dots, v_{11}\rangle_M$, and L the subset of edges of M with one endpoint in N and the other in H. Adding degrees we get

$$2|E(H)| + |L| = 30, \quad 2|E(N)| + |L| = 26;$$

thus $|E(H)| = |E(N)| + 2$. Note that $|E(H)| \le 8$, since if $|E(H)| = 10$, contracting the edges of $N_M[v_1]$ produces a K_6 minor of M; if $|E(H)| = 9$, that is, $H \simeq K_5^-$, then, without loss of generality, assume $v_7v_8 \notin E(H)$. Since $\deg_M(v_7) = \deg_M(v_8) = 6$, it follows that both v_7 and v_8 have each three neighbors among the five vertices of N; thus they share a common neighbor in N, say v_i. Contracting the edges v_iv_7 and v_1v_j, for $2 \le j \ne i \le 6$, we obtain a K_6 minor of M.

We may then assume that $|E(H)| \le 8$, and thus $|E(N)| \le 6$. If any vertex v_i of N has at most one neighbor in N, contracting the edge v_1v_i would produce a minor of M of order 10 and size at least 31. By Theorem 4, this would contain a K_6 minor. So every vertex of N has at least two neighbors in N. If follows that $5 \le |E(N)| \le 6$ and, by [Read and Wilson 1998], N is isomorphic to one of the four graphs in Figure 4.

If $\deg_N(v_2) = 2$, then contracting the edge v_1v_2 produces a minor of M of order 10 and size 30, with degree sequence $(5, 6, 6, 6, 6, 6, 6, 6, 6, 7)$. The following lemma shows that M contains a K_6 minor.

Lemma 10. *Let M denote a graph of order 10 and size 30 such that $\delta(M) \ge 5$. Assume that M is not apex and has at most five vertices of degree 5. Then M contains a K_6 minor.*

Proof. By Theorem 5, either M contains a K_6 minor or is an MP_1-cockade. Since M is not apex, it follows that M is the clique sum over $S \simeq K_4$ of two MP_1-cockades. Since $\delta(M) \ge 5$, any connected component of $M - S$ is of size at least 2. If any of the connected components of $M-S$ has exactly two vertices, then that component together with S induces a K_6 subgraph of M. The only situation left to discuss is when $M-S$ has exactly two size-3 connected components, Q_1 and Q_2. At least one of them, say Q_1, contains at most two vertices of degree 5. If we denote by L' the set of edges connecting the vertices of Q_1 to the vertices of S, then

$$2|E(Q_1)|+|L'|\geq 6+5+5=16.$$

Hence

$$|E(Q_1)|+|L'|\geq 13 \quad\Longrightarrow\quad |E(Q_1)|+|L'|+|E(S)|\geq 19;$$

thus Q_1 and S induce a subgraph of M of order 7 and size 19. By Theorem 4, this subgraph contains a K_6 minor. □

If $\deg_N(v_2)=3$, then N is isomorphic to either graph B or graph C in Figure 4. Furthermore, v_2 neighbors in N a vertex (say v_3) of total degree 6 which has degree 2 in N and does not neighbor the vertex of degree 8. Contracting the edge v_1v_3 produces a minor P of M of order 10 and size 30. Furthermore, the degree sequence of this minor is (4, 5, 6, 6, 6, 6, 6, 6, 7, 8) and thus it is not apex. By Theorem 5, P either contains a K_6 minor or it is the clique sum over $S\simeq K_4$ of two MP_1-cockades. Every vertex of S has degree at least 5 since it must connect to every connected component of $P-S$. Let Q_1 denote the connected component of $P-S$ which contains the vertex of degree 4. Let H denote the graph induced by the vertices of $P-(S\cup Q_1)$ and let L'' denote the set of edges of P with one endpoint in S and the other in H.

If $|V(Q_1)|=1$, then

$$|L''|+|E(H)|=20,$$
$$|L''|+2|E(H)|\geq 6+6+6+6+5=29.$$

It follows that $|E(H)|\geq 9$; that is, $H\simeq K_5^-$ or $H\simeq K_5$. For $H\simeq K_5$, as every vertex of H is adjacent to at least a vertex of S, contracting S produces a K_6 minor of P. If $H\simeq K_5^-$, assume $a,b\in V(H)$ and $ab\notin E(H)$. If $\deg_P(a)=5$ or $\deg_P(b)=5$, then a and b share at least one common neighbor s in S. Contracting the edge sa and then contracting the edges of the graph induced by the vertices of $Q_1\cup S-\{s\}$ produces a K_6 minor of P, as every vertex of H other than a or b has degree at least 6 and must therefore be adjacent to at least two vertices of S. Finally, if $\deg_P(a)\geq 6$ and $\deg_P(b)\geq 6$, then a and b share at least two neighbors among the vertices of S, say s_1 and s_2. Since $V(H)\backslash\{a,b\}$ contains at most one vertex of degree 5, that vertex is adjacent to at most one of $\{s_1,s_2\}$, say s_1. Contracting the edge s_2a and then contracting the edges of the graph induced by the vertices of $Q_1\cup S-\{s_2\}$ produces a K_6 minor of P.

If $|V(Q_1)| = 2$, then $\langle Q_1 \cup S\rangle_P \simeq K_6^-$ and

$$|L''| + |E(H)| = 16,$$
$$|L''| + 2|E(H)| \geq 6+6+6+6 = 24.$$

It follows that $|E(H)| \geq 8$, which is a contradiction (H has only four vertices). It follows that $|V(Q_1)| = 3$ and that $\langle S \cup H\rangle_P$ contains a K_7^- minor.

If $\deg_N(v_2) = 4$, then $N \simeq D$ of Figure 4 and contracting the edge connecting v_1 to the common neighbor of v_2 and v_6 in N produces a minor P of M of order 10 and size 30, with degree sequence (4, 6, 6, 6, 6, 6, 6, 6, 7, 7), where the two degree-7 vertices and the degree-4 vertex form a triangle. Theorem 5 shows that P is a clique sum over $S \simeq K_4$ of two MP_1-cockades. Furthermore $P - S$ has exactly two connected components, Q_1 and Q_2. As any vertex that's part of the clique has at least degree 5 in P, we may assume that the vertex of degree 4 is a vertex of Q_1. Unless $|Q_1| = 1$, both Q_1 and Q_2 will contain vertices of degree at least 6 in P; hence $|Q_1| = |Q_2| = 3$. But this implies that contracting any edge incident to the vertex of degree 4 in Q_1 produces a K_6 minor of the graph induced by $Q_1 \sqcup S$.

If $|V(Q_1)| = 1$, then let L'' denote the set of edges in P with one endpoint in K_4 and the other in Q_2. It follows that

$$|L''| = 7+7+6+6-12-4 = 10,$$
$$|E(Q_2)| = 10;$$

hence $Q_2 \simeq K_5$ and thus contracting the edges of the subgraph induced by $Q_1 \sqcup K_4$ produces a K_6 minor.

Subcase 3.2: Assume that $v_3v_4 \notin E(G)$. Contracting the edge v_1v_2 produces a minor M of G of order 11 and size 33. Furthermore, the degree sequence of this minor is (5, 5, 6, 6, 6, 6, 6, 6, 6, 6, 8). Via a relabeling, for the rest of this subcase, we may assume that $\deg_M(v_1) = \deg_M(v_7) = 5$, and that $\deg_M(v_6) = 8$. As before, let $N = \langle v_2, v_3, v_4, v_5, v_6\rangle_M$ be the subgraph of M induced by all the neighbors of v_1, $H = \langle v_7, \ldots, v_{11}\rangle_M$, and L the subset of edges of M with one endpoint in N and the other in H. Adding degrees we get

$$2|E(H)| + |L| = 29, \quad 2|E(N)| + |L| = 27;$$

thus $|E(H)| = |E(N)|+1$. Furthermore, if $|E(H)| = 10$, that is, $H \simeq K_5$, contracting v_1v_i for $2 \leq i \leq 6$ produces a K_6 minor.

Assume $|E(H)| = 9$ and $|E(N)| = 8$. Then by [Read and Wilson 1998], N is isomorphic to one of the two graphs in Figure 5.

If $N \simeq A$ in Figure 5, as all vertices of N have minimum degree 6, contracting the vertices of H (which is connected) to a single point and then further contracting

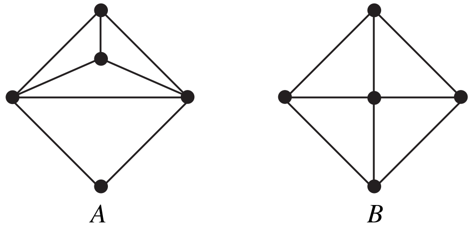

Figure 5. Graphs of order 5 and size 8, with minimum degree 2.

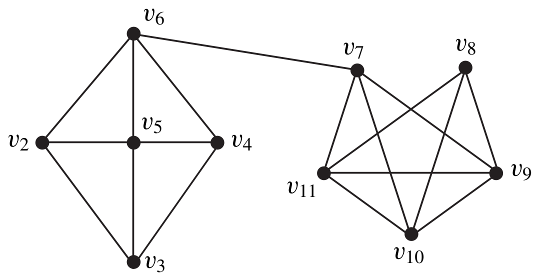

Figure 6. Subcase 3.2.1: $\deg_N(v_6) = 3$ and $\deg_H(v_7) = 3$.

the edge joining the newly obtained point and the only vertex of degree 2 in N produces a K_6 minor.

If $N \simeq B$ in Figure 5, then we distinguish four cases based on the position of v_6 and v_7 inside N and H, respectively.

Subcase 3.2.1: Assume $\deg_N(v_6) = 3$ and $\deg_H(v_7) = 3$; see Figure 6. Without loss of generality, assume $\deg_H(v_8) = 3$; that is, v_7v_8 is the only edge missing in the complete graph on the vertices of H. If v_8 neighbors v_6, contracting the edge v_6v_8 and all the edges of $\langle v_1, v_2, v_3, v_4, v_5\rangle_M$ produces a K_6 minor of M. If v_6 does not neighbor v_8, then contracting the edges of $\langle v_1, v_2, v_3, v_4, v_5, v_8\rangle_M$ produces a K_6 minor of M.

Subcase 3.2.2: Assume $\deg_N(v_6) = 4$ and $\deg_H(v_7) = 3$; see Figure 7. Without loss of generality, we may assume $\deg_H(v_8) = 3$. If v_7 and v_8 share a neighbor in N, say v_j, then contracting v_jv_7 and then all the edges of $\langle v_1, N - v_j\rangle_M$, we obtain

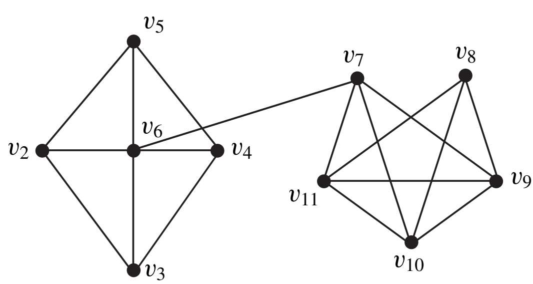

Figure 7. Subcase 3.2.2: $\deg_N(v_6) = 4$ and $\deg_H(v_7) = 3$.

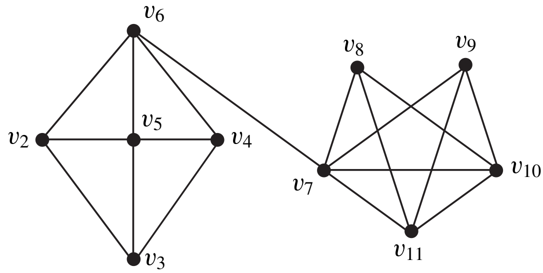

Figure 8. Subcase 3.2.3: $\deg_N(v_6) = 3$ and $\deg_H(v_7) = 4$.

a K_6 minor of G. So, as a set, $\{v_7, v_8\}$ neighbors all the vertices of N. Without loss of generality, $v_5v_7, v_6v_7, v_5v_9 \in E(M)$, $v_3v_5 \notin E(N)$. If v_3 neighbors either of v_{10} or v_{11}, say v_{10}, then contracting the edges v_3v_{10}, v_5v_9 will connect v_3 and v_5, contracting all the edges of $\langle v_2, v_7, v_8, v_{11}\rangle_M$ will connect v_2 and v_4, and thus we obtain a K_6 minor. But then, v_3 must neighbor v_9, and contracting the edge v_3v_9 and all the edges of $\langle v_7, v_8, v_{10}, v_{11}, v_2\rangle_M$ produces a K_6 minor.

Subcase 3.2.3: Assume $\deg_N(v_6) = 3$ and $\deg_H(v_7) = 4$; see Figure 8. Without loss of generality, we may assume $\deg_H(v_8) = \deg_H(v_9) = 3$ and $\deg_N(v_5) = 4$. Since v_8 and v_9 each connect to three vertices of N, they have a common vertex in N. If this vertex is not v_6, say v_i, then contracting the edges v_iv_8 and v_1v_j for $2 \le j \ne i \le 6$, we obtain a K_6 minor of M. So $\{v_7, v_8, v_9\}$, as a set, neighbors all the vertices of N. Since v_3 does not neighbor v_7 and cannot neighbor both v_8 and v_9, it must neighbor one of v_{10} or v_{11}. But then, contracting the edges of $\langle v_6, v_{10}, v_{11}\rangle_M$ and then contracting the edges of $\langle v_7, v_8, v_9, v_2\rangle_M$, we obtain a K_6 minor of M.

Subcase 3.2.4: Assume $\deg_N(v_6) = 4$ and $\deg_H(v_7) = 4$; see Figure 9. Without loss of generality, we may assume that $\deg_H(v_8) = \deg_H(v_9) = 3$. Since both v_8 and v_9 connect to three vertices of N, they must share at least one common neighbor in N. If that common neighbor is not v_6, say v_2, contracting the edges v_2v_8, and v_1v_i for $3 \le i \le 6$, we obtain a K_6 minor of M. It follows that v_6 neighbors v_7, v_8 and v_9 and that, as a set, $\{v_8, v_9\}$ neighbors all the vertices of N. If $\{v_{10}, v_{11}\}$ neighbors,

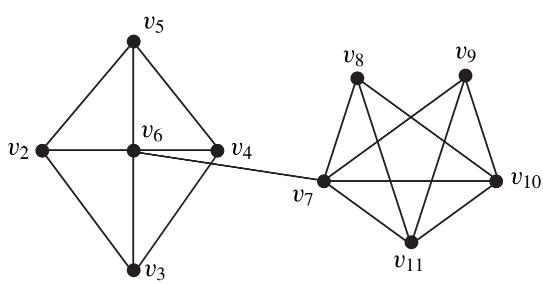

Figure 9. Subcase 3.2.4: $\deg_N(v_6) = 4$ and $\deg_H(v_7) = 4$.

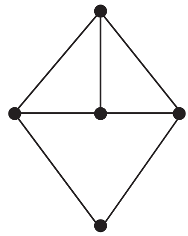

Figure 10. The unique graph of order 5, size 7, and minimum degree 2, with exactly one vertex of degree 2.

as a set, any nonneighbors in N, say v_2 and v_4, then contracting the edges $v_{10}v_{11}$, v_2v_{10}, and all the edges of $\langle v_5, v_7, v_8, v_9\rangle_M$, we obtain a K_6 minor of M. If not, as v_6 neighbors neither v_{10} nor v_{11}, we know that v_{10} and v_{11} must both neighbor an edge of N not incident to v_6, say v_2v_3. But since v_2 neighbors v_1, v_3, v_5, v_6, v_{10} and v_{11} and one of v_8 or v_9, it follows that v_2 has degree 7 in M, a contradiction.

It follows that $|E(N)| \leq 7$. If any of $v_2, \ldots, v_6$ have degree at most 1 in N, contracting the edge connecting that vertex to v_1 would produce minor of M of order 10 and size at least 31, which would have a K_6 minor by Theorem 4. This shows that $\delta(N) \geq 2$ and that $|E(N)| \geq 5$. Furthermore, if any of the degree-6 neighbors of v_1 have degree 2 in N, contracting the edge connecting that neighbor to v_1 would produce a nonapex graph of order 10 and size 30, with minimal degree at least 5, and at most three vertices of degree 5. By Lemma 10, this graph would contain a K_6 minor. This observation handles the cases $|E(N)| = 5$ and $|E(N)| = 6$, since any graph on five vertices with minimum degree 2 and size at most 6 has at least two vertices of degree exactly 2.

Assume that $|E(N)| = 7$, $\delta(N) = 2$, and N has only one vertex of degree 2. Then N is isomorphic to the graph in Figure 10. Furthermore, $\deg_N(v_6)$ is 2; thus v_6 neighbors all the vertices of H. If v_7 neighbors two or more vertices of N, then contracting v_1v_i for $2 \leq i \leq 5$ and H to one of its K_4 minors (H has 8 edges and 5 vertices, by Theorem 4 it has a K_4 minor) we obtain a K_6 minor. It follows that v_7 connects to v_8, v_9, v_{10} and v_{11}. Furthermore, the open neighborhood of v_7 contains exactly eight edges. By symmetry between v_1 and v_7 and Subcase 3.2, $|E(N)| = 8$, it follows that M has a K_6 minor. □

3. Future explorations

(1) Is it true that any simple graph of order at most 14 and minimum degree at least 6, which is not apex, contains a K_6-minor? Note that in the proof of Theorem 7, we used weaker versions of Lemmas 9 and 10. Similar lemmas hold for graphs of orders 13 and 12, respectively. They provide a first step in generalizing Theorem 7 for graphs of order at most 14.

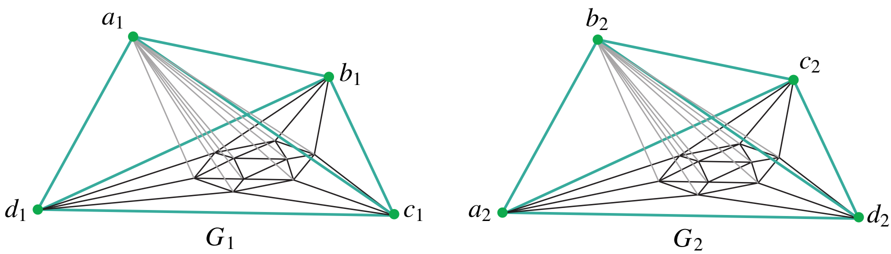

Figure 11. Graphs G_1 and G_2 are identified along the highlighted tetrahedra to obtain the graph G.

(2) The result of this paper shows that, for graphs of order 12, weaker assumptions are needed for the conclusion of Jørgensen's conjecture to be true. What is the minimum $n > 12$ for which the condition of minimum degree 6 is no longer sufficient and the 6-connected condition is needed? Such n would have to be at most 22, as the following example demonstrates.

Let $G_1 \simeq G_2 \simeq K_1 * \text{Ic}$, where Ic denotes the icosahedral graph (5-regular, maximal planar, order 12). Let G denote a clique sum over K_4 of G_1 and G_2, done in such a way that the cones are not identified to each other (so that the maximum degree of G is 15). In Figure 11, a_1 and a_2, b_1 and b_2, c_1 and c_2, and d_1 and d_2 are respectively identified. Then $\delta(G) = 6$, G is not apex, and it has no K_6 minor.

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