involve
a journal of mathematics

Eta-quotients of prime or semiprime level and elliptic curves

Michael Allen, Nicholas Anderson,
Asimina Hamakiotes, Ben Oltsik and Holly Swisher

msp

2020 vol. 13, no. 5

INVOLVE 13:5 (2020)
https://doi.org/10.2140/involve.2020.13.879

Eta-quotients of prime or semiprime level and elliptic curves

Michael Allen, Nicholas Anderson,
Asimina Hamakiotes, Ben Oltsik and Holly Swisher

(Communicated by Ken Ono)

From the modularity theorem proven by Wiles, Taylor, Conrad, Diamond, and Breuil, we know that all elliptic curves are modular. It has been shown by Martin and Ono exactly which are represented by eta-quotients, and some examples of elliptic curves represented by modular forms that are linear combinations of eta-quotients have been given by Pathakjee, RosnBrick, and Yoong.

In this paper, we first show that eta-quotients which are modular for any congruence subgroup of level N coprime to 6 can be viewed as modular for $\Gamma_0(N)$. We then categorize when even-weight eta-quotients can exist in $M_k(\Gamma_1(p))$ and $M_k(\Gamma_1(pq))$ for distinct primes p, q. We conclude by providing some new examples of elliptic curves whose corresponding modular forms can be written as a linear combination of eta-quotients, and describe an algorithmic method for finding additional examples.

1. Introduction and statement of results

Dedekind's eta-function $\eta(\tau)$, defined for $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ by

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q := e^{2\pi i \tau}$, is arguably the most well-known half-integral weight modular form. Its modular transformation properties for a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ are given by

$$\eta(A\tau) = v(A)(c\tau + d)^{1/2} \eta(\tau), \tag{1}$$

where

$$v(A) := \begin{cases} (d/|c|)\, e^{(i\pi/12)((a+d)c - bd(c^2-1) - 3c)} & \text{if } c \equiv 1 \pmod 2, \\ (c/|d|)\, e^{(i\pi/12)((a+d)c - bd(c^2-1) + 3d - 3 - 3cd)} & \text{if } c \equiv 0 \pmod 2. \end{cases} \tag{2}$$

MSC2020: primary 11F20, 11F37; secondary 11G05.
Keywords: eta-quotients, modular forms, elliptic curves.
This work was supported by NSF grant DMS-1359173.

Dedekind's eta-function has been featured prominently in work motivated by Ramanujan's study of the partition counting function $p(n)$ (see [Ono 2000; Ahlgren and Ono 2001] for example), as well as in the representation theory of the monster group, studied by Conway and Norton [1979], Borcherds [1992], and others. Due to its expression as a simple infinite product, it is easy to compute expansions numerically. Thus it is useful when a modular form can be expressed in terms of products or quotients of $\eta(\tau)$. By *eta-quotient of level N and integer weight k_f* we mean a function of the form

$$f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta},$$

where N is a positive integer, $r_\delta \in \mathbb{Z}$, and $k_f = \frac{1}{2}\sum r_\delta \in \mathbb{Z}$.

Work on various classifications of eta-quotients has been of interest; see in particular [Dummit, Kisilevsky and McKay 1985; Martin 1996; Lemke Oliver 2013]. Moreover, the famous works [Wiles 1995; Taylor and Wiles 1995] proving Fermat's last theorem, and in particular the Shimura–Taniyama conjecture, showed that elliptic curves were attached to modular forms, and thus the question of when these modular forms can be expressed in terms of eta-quotients is a natural one.

Theorem 1.1 (modularity theorem [Diamond and Shurman 2005]). *Every elliptic curve E over $\mathbb{Q}$ with conductor N has an L-function*

$$L(E, s) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s}$$

such that the Fourier series

$$f(\tau) = \sum_{n=1}^{\infty} a_E(n) q^n, \quad q = e^{2\pi i \tau},\ \tau \in \mathbb{H},$$

represents a level-N cusp form of weight 2.

In light of Theorem 1.1, Martin and Ono [1997] classified all eta-quotients which are weight-2 newforms, as well as their associated elliptic curves. It is natural to ask when elliptic curves are associated to modular forms that can be written as linear combinations of eta-quotients. Recently, Pathakjee, RosnBrick, and Yoong [2012] demonstrated four such examples, utilizing spaces of cusp forms which are spanned by eta-quotients. Further work on when spaces of modular forms are spanned by eta-quotients has been done in [Rouse and Webb 2015; Arnold-Roksandich, James and Keaton 2018a; Kilford 2007], for example.

Given a congruence subgroup group $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, we use the notation $S_k(\Gamma)$, $M_k(\Gamma)$, and $M_k^!(\Gamma)$ to denote the complex vector spaces of weight-k cusp forms, holomorphic modular forms, and weakly holomorphic modular forms, respectively. When $\Gamma = \Gamma_0(N)$ for a positive integer N, we write $S_k(\Gamma_0(N), \chi)$, $M_k(\Gamma_0(N), \chi)$,

and $M_k^!(\Gamma_0(N), \chi)$ to denote the spaces of weight-k cusp forms, holomorphic modular forms, and weakly holomorphic modular forms, respectively, with Nebentypus χ.

In this paper, we first show that eta-quotients which are modular for any congruence subgroup of level N can be viewed as modular for $\Gamma_0(N)$. We then categorize when eta-quotients of even weight can exist in $M_k(\Gamma_1(p))$ and $M_k(\Gamma_1(pq))$ for distinct primes p, q. We conclude by providing some new examples of elliptic curves whose corresponding modular forms can be written as a linear combination of eta-quotients, and describe an algorithmic method for finding additional examples.

1A. ***Viewing eta-quotients over*** $\boldsymbol{\Gamma_0(N)}$**.** We first review a few key theorems about the modularity of eta-quotients and their orders of vanishing at cusps.

The following well-known theorem originating in [Newman 1957; 1959; Gordon and Hughes 1993], provides explicit modularity properties with respect to $\Gamma_0(N)$ for eta-quotients of specific shapes.

Theorem 1.2 [Ono 2004, Theorem 1.64]. *Let f be the eta-quotient of level N given by*

$$f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta}.$$

If f satisfies both

$$\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24} \quad \text{and} \quad \sum_{\delta \mid N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then for $k = \frac{1}{2}\sum_{\delta \mid N} r_\delta \in \mathbb{N}$ and $\chi(d) = ((-1)^k s/d)$, where $s = \prod_{\delta \mid N} \delta^{r_\delta}$, we have $f \in M_k^!(\Gamma_0(N), \chi)$, i.e., for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$,

$$f(M\tau) = \chi(d)(c\tau + d)^k f(\tau).$$

Remark 1.3. In the case where $\gcd(N, 6) = 1$, the two conditions above are equivalent. This is because any $\delta \mid N$ must satisfy $\gcd(\delta, 24) = 1$, and hence δ is its own inverse modulo 24. Thus,

$$\sum_{\delta \mid N} \frac{N}{\delta} r_\delta \equiv N \sum_{\delta \mid N} \delta r_\delta \pmod{24}.$$

As $\gcd(N, 24) = 1$, N is not a zero divisor in $\mathbb{Z}/24\mathbb{Z}$, so

$$N \sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24} \quad \text{if and only if} \quad \sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24}.$$

The next theorem originating in [Ligozat 1974], and further appearing in [Biagioli 1990; Martin 1996], provides a mechanism for calculating the relative orders of vanishing at cusps for eta-quotients satisfying Theorem 1.2.

Recall that if h is the width of the cusp r with respect to the group Γ, then the *order of vanishing relative to the group* Γ of a modular form f at the cusp r is given by $v_\Gamma(f, r) = h \cdot \mathrm{inv}_r(f)$, where $\mathrm{inv}_r(f)$ is the invariant order of vanishing of f at r and is always integral. We note that for the cusp at infinity, we always have $v_\Gamma(f, r) = \mathrm{inv}_r(f)$.

Theorem 1.4 [Ono 2004, Theorem 1.65]. *Let c, d, and N be positive integers with $d \mid N$ and $\gcd(c, d) = 1$. Then if $f(\tau)$ is an eta-quotient satisfying the conditions given in Theorem 1.2 for N, then the order of vanishing for $f(\tau)$ at the cusp c/d relative to $\Gamma_0(N)$ is*

$$v_{\frac{c}{d}} := v_{\Gamma_0(N)}\left(f, \frac{c}{d}\right) = \frac{N}{24} \sum_{\delta \mid N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, N/d) d \delta}.$$

Calculating orders of vanishing of modular forms at cusps can be extremely useful, due to the following result known as Sturm's bound.

Theorem 1.5 (Sturm's bound [Murty, Dewar and Graves 2015]). *Let Γ be a congruence subgroup and $f \in M_k(\Gamma)$. Let $r_1, \ldots, r_t$ be the Γ-inequivalent cusps of Γ. If*

$$\sum_{i=1}^{t} \mathrm{inv}_{r_i}(f) > \frac{k[\mathrm{SL}_2(\mathbb{Z}) : \{\pm I\}\Gamma]}{12},$$

then $f = 0$.

Remark 1.6. We note that Theorem 1.5 provides a direct way to check if two modular forms are equal by considering their q-expansions. Namely, if $f, g \in M_k(\Gamma)$ and their Fourier expansions at $i\infty$ agree to a power of q past the bound in Theorem 1.5, then they must be equal.

Our first theorem shows that when $\gcd(N, 6) = 1$, any eta-quotient which is modular for a congruence subgroup of level N is in fact modular for $\Gamma_0(N)$ for some Nebentypus character χ. We note that the hypothesis that $\gcd(N, 6) = 1$ is necessary; for example, one can show that $\eta(\tau)^2$ is a weight-1 modular form for $\Gamma(12)$.[1]

Remark 1.7. Let N be a positive integer with $\gcd(N, 6) = 1$ and suppose

$$f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta} \in M^!_{k_f}(\Gamma(N)).$$

Then since $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma(N)$, it follows that f has a Fourier expansion that is a power series in $q^{1/N}$. Of course, as an eta-quotient f can also be expressed as a power series in $q^{1/24}$, in particular with first term $q^{1/24 \sum \delta r_\delta}$. Since $\gcd(N, 24) = 1$,

[1] See https://mathoverflow.net/questions/297049/is-eta-tau2-a-modular-form-of-weight-1-on-gamma12/297053, where Jeremy Rouse describes why this is true.

it follows that $\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24}$. Thus by Theorem 1.2 and Remark 1.3 we can conclude that $f(\tau) \in M^!_{k_f}(\Gamma_0(N), \chi)$, where χ is defined as in Theorem 1.2.

Remark 1.8. Let N be a positive integer with $\gcd(N, 6) = 1$. Using that fact that $M^!_k(\Gamma_0(N), \chi) \subset M^!_k(\Gamma_1(N))$ as well as Remarks 1.7 and 1.3, it follows that $f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta} \in M^!_{k_f}(\Gamma_1(N))$ if and only if $f(\tau)$ satisfies the conditions in Theorem 1.2.

From Remark 1.8, we see that despite there being many more cusps of $\Gamma_1(N)$ than $\Gamma_0(N)$, we only need to consider the cusps of $\Gamma_0(N)$ when calculating orders of vanishing of eta-quotients in $M^!_k(\Gamma_1(N))$. In [Martin 1996], a complete set of representatives for the cusps of $\Gamma_0(N)$ is given by

$$C_{\Gamma_0(N)} = \left\{ \frac{a_c}{c} \in \mathbb{Q} : c \mid N,\ 1 \le a_c \le N,\ \gcd(a_c, N) = 1 \right. \\ \left. \text{and } a_c \equiv a'_c \pmod{\gcd(c, N/c)} \text{ if and only if } a_c = a'_c \right\}.$$

In the case where N is square-free, $\gcd(c, N/c) = 1$ for all c which divide N. This gives the following complete set of representatives:

$$C_{\Gamma_0(N)} = \left\{ \frac{1}{d} \in \mathbb{Q} : d \mid N \right\}. \tag{3}$$

1B. ***Classifications.*** Our first results deal with classifying when eta-quotients can exist in spaces of holomorphic modular forms of even weight and prime or semiprime level. The following theorem uses techniques from [Arnold-Roksandich, James and Keaton 2018a] and also addresses an error in Corollary 3.2 of that paper.

Theorem 1.9. *Let $p \ge 5$ be prime, set $h = \frac{1}{2}\gcd(p-1, 24)$, and let k be an even integer. Then there exists $f = \eta(\tau)^{r_1}\eta(p\tau)^{r_p} \in M_k(\Gamma_1(p))$ if and only if both of the following conditions hold*:

(1) $h \mid k$.

(2) *It is not the case that* $p \ne 5,\ p \equiv 5 \pmod{24}$, *and* $k = 2$.

We also prove a theorem of similar flavor for eta-quotients of semiprime level, which extends [Arnold-Roksandich, James and Keaton 2018b, Theorem 5.8].

Theorem 1.10. *Let $p, q \ge 5$ be distinct primes, $N = pq$, and k an even integer. Let $h = \frac{1}{2}\gcd(24, p-1, q-1)$. Then there exists $f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta} \in M_k(\Gamma_1(pq))$ if and only if both of the following conditions hold*:

(1) $h \mid k$.

(2) *It is not the case that* $(p, q) \pmod{24} \in \{(1, 5), (5, 1), (5, 5)\}$, $p > 5$ *and* $q > 5$, *and* $k = 2$.

1C. ***Writing modular forms attached to elliptic curves in terms of eta-quotients.*** Utilizing our work in Section 3, we explore when $S_2(\Gamma_0(pq))$ has a basis consisting of eta-quotients. This allows us to provide examples of elliptic curves attached via the modularity theorem (Theorem 1.1) to linear combinations of eta-quotients.

Theorem 1.11. *Let E be an elliptic curve with conductor 35. Then E is associated via the modularity theorem to the modular form $f(\tau) \in S_2(\Gamma_0(35))$ given by*

$$f(\tau) = \eta(\tau)^2\eta(35\tau)^2 + \eta(5\tau)^2\eta(7\tau)^2.$$

We are also able to provide an example even when $S_2(\Gamma_0(pq))$ does not have a basis consisting of eta-quotients.

Theorem 1.12. *Let E be an elliptic curve with conductor 55. Then E is associated via the modularity theorem to the modular form $f(\tau) \in S_2(\Gamma_0(55))$ given by*

$$f(\tau) = \sum_{i=1}^{40} c_i \frac{g_i(\tau)}{a(\tau)},$$

where in Section 4 the coefficients c_i are given in Table 2, and the eta-quotients $g_i(\tau)/a(\tau)$ are given in Table 3. The first three $g_i(\tau)/a(\tau)$, $i = 1, 2, 3$, are holomorphic modular forms.

1D. ***Outline of the rest of the paper.*** The rest of the paper is devoted to proving our results, and surrounding discussions. In Section 2, we prove Theorem 1.9, and in Section 3, we prove Theorem 1.10, discussing also square-free level cases. In Section 4, we prove Theorems 1.11 and 1.12.

2. Eta-quotients of prime level

In this section our goal is to prove Theorem 1.9, which classifies when eta-quotients can exist in $M_k^!(\Gamma_1(p))$ for p prime. And of course by Remark 1.7 we know that eta-quotients in $M_k^!(\Gamma_1(p))$ are actually in $M_k^!(\Gamma_0(p), \chi)$ for some character χ. By (3), there are two cusps of $\Gamma_0(p)$: 1 and $1/p$. We first state the following useful result of [Arnold-Roksandich, James and Keaton 2018a], but offer an alternative proof which we will see in Section 3 generalizes to square-free levels.

Lemma 2.1 [Arnold-Roksandich, James and Keaton 2018a, Theorem 1.2]. *Fix a prime $p \geq 5$ and let $h = \frac{1}{2}\gcd(p-1, 24)$. There exists $f(\tau) = \eta(\tau)^{r_1}\eta(p\tau)^{r_p}$ in $M_k^!(\Gamma_1(p))$ if and only if $h \mid k$.*

Proof. Suppose there exists $f(\tau) = \eta(\tau)^{r_1}\eta(p\tau)^{r_p}$ in $M_k^!(\Gamma_1(p))$. By Remark 1.8 we see that f satisfies the conditions in Theorem 1.4, and thus

$$v_1 = \tfrac{1}{24}(pr_1 + r_p). \tag{4}$$

Since $k = \frac{1}{2}(r_1 + r_p)$, (4) is equivalent to

$$24v_1 - (p-1)r_1 = 2k. \tag{5}$$

Equation (5) can be viewed as a linear Diophantine equation in variables v_1 and r_1. In this light, the existence of a solution implies $\gcd(p-1, 24) \mid 2k$. As $p \neq 2$, we know $p-1$ is even and so $2 \mid \gcd(24, p-1)$. Therefore, $h = \frac{1}{2}\gcd(p-1, 24)$ is an integer and $h \mid k$.

Conversely, if $h \mid k$, then there exist integers v_1, r_1 which give a solution to (5). Plugging these into (4) gives an integer r_p. From (4) we see the first condition of Theorem 1.2 is satisfied, and so, since $(p, 6) = 1$, Remark 1.3 implies that $f(\tau) \in M_k^!(\Gamma_1(p))$. □

Our proof of Theorem 1.9 will also rely on the following result of [Rouse and Webb 2015].

Theorem 2.2 [Rouse and Webb 2015, Theorem 2]. *Suppose*

$$f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta} \in M_{k_f}(\Gamma_0(N)).$$

Then we have

$$\sum_{\delta \mid N} |r_\delta| \leq 2k \prod_{p \mid N} \left(\frac{p+1}{p-1}\right)^{\min\{2, \mathrm{ord}_p(N)\}},$$

where $\mathrm{ord}_p(N)$ *denotes the* p*-adic valuation of* N*.*

Remark 2.3. The proof of this theorem carries through for spaces $M_k(\Gamma_0(N), \chi)$ with character. Thus in light of Remark 1.8, Theorem 2.2 holds for any eta-quotient in $M_k(\Gamma_1(N))$.

We also require two additional lemmas which will allow us to deal with certain cases in the proof of Theorem 1.9. These lemmas both deal with the value $k(p+1)/12$, where $p \geq 5$ is prime, and k is an even integer such that $h = \frac{1}{2}\gcd(p-1, 24) \mid k$. Note that in this case $k(p+1)/12$ must be an integer, since if 3 doesn't divide $p+1$, then 3 must divide $p-1$, and so 3 divides h and thus k.

Lemma 2.4. *Fix a prime* $p \geq 5$*, and let* k *be a positive even integer. If* $k(p+1)/12$ *is even, then there exists an eta-quotient in* $M_k(\Gamma_1(p))$*.*

Proof. Consider the eta-quotient

$$f(\tau) = \eta(\tau)^k \eta(p\tau)^k.$$

Since $k(p+1)/12$ is even, and by Remark 1.3, we see that $f(\tau)$ satisfies the conditions of Theorem 1.2, and so $f(\tau) \in M_k^!(\Gamma_1(p))$. Moreover, by Theorem 1.4,

$$v_1 = v_{1/p} = \tfrac{1}{24}k(p+1), \tag{6}$$

which is a positive integer. Thus $f(\tau) \in M_k(\Gamma_1(p))$. □

Remark 2.5. Note that we did not need the assumption that $h \mid k$ in the previous lemma. This is because when $k(p+1)/12$ is even, we must have $h \mid k$. To see this, observe that if $k(p+1)/12 = 2n$, then $2k = 24n - k(p-1)$, and so $\gcd(24,\ p-1) \mid 2k$.

The last lemma is a simple divisibility argument.

Lemma 2.6. *Let p be prime, $h = \frac{1}{2}\gcd(p-1, 24)$, and k be an even integer. If $h \mid k$ and $k(p+1)/12$ is odd, then $(p-1)/(2h)$ is odd.*

Proof. We first note that if $p = 2$, then $(p-1)/(2h) = 1$ and so is odd regardless of k. We now let $p \geq 3$. Suppose $(p-1)/(2h)$ is even. By the definition of h, $(p-1)/(2h)$ and $12/h$ are relatively prime, so $12/h$ must be odd. But then 12 is an odd integer times h, so we have that $4 \mid h$, and so $4 \mid k$. Thus $8 \mid k(p+1)$, since p is odd. Therefore $k(p+1)/12$ is even, which contradicts our assumptions. □

We are now able to prove Theorem 1.9.

Proof of Theorem 1.9. First, we assume that there exists

$$f(\tau) = \eta(\tau)^{r_1}\eta(p\tau)^{r_p} \in M_k(\Gamma_1(p)).$$

By Lemma 2.1, we have that $h \mid k$. To complete this direction of the proof, we show that there are no eta-quotients in $M_2(\Gamma_1(p))$ when $p \equiv 5 \pmod{24}$ and $p > 5$. To do this, we recall Remark 2.3, and employ Theorem 2.2. Fix $p \equiv 5 \pmod{24}$ with $p > 5$, and suppose

$$f(\tau) = \eta(\tau)^{r_1}\eta(p\tau)^{r_p} \in M_2(\Gamma_1(p)).$$

Theorem 2.2 gives us that

$$|r_1| + |r_p| \leq 4\left(\frac{p+1}{p-1}\right).$$

This upper bound decreases as p increases, and so will be largest when $p = 29$, which gives a bound less than 5. Since r_1 and r_p are integers, we have

$$|r_1| + |r_p| \leq 4. \tag{7}$$

Moreover, by Theorem 1.4 we have

$$24v_1 = pr_1 + r_p,$$

$$24v_{1/p} = r_1 + pr_p.$$

Since $f(\tau) \in M_k(\Gamma_1(p))$, it must be that $v_1,\ v_{1/p} \geq 0$. This guarantees that $r_1,\ r_p \geq 0$. Namely, if $r_1 < 0$, then using (7), we see that

$$24v_{1/p} \leq -29|r_1| + |r_p| < 0,$$

which contradicts that $v_{1/p} \geq 0$. Similarly, $r_p < 0$ implies that $v_1 < 0$.

Since $p \equiv 5 \pmod{24}$, Remark 1.8 implies that

$$r_1 + 5r_p \equiv 0 \pmod{24}. \tag{8}$$

However, there is no pair of nonnegative integers r_1, r_p, not both zero, that satisfy both (7) and (8). Thus we have a contradiction and so no such eta-quotient $f(\tau)$ can exist.

We now prove the converse by construction. Suppose that $p \geq 5$ is a prime and k is an even integer such that $h = \frac{1}{2}\gcd(p-1, 24) \mid k$. We construct an eta-quotient in $M_k(\Gamma_1(p))$ for every such case of p and k except when $k = 2$ and $p \equiv 5 \pmod{24}$, with $p > 5$.

We first consider the case where $p \not\equiv 5 \pmod 8$. By Lemma 2.4, it suffices to prove that $k(p+1)/12$ is even, which, since $k(p+1)/12$ is an integer, means showing that $8 \mid k(p+1)$. We already know that both k and $p+1$ are even, so it suffices to show that either k or $p+1$ is divisible by 4. If $p \equiv 3 \pmod 4$, then $4 \mid p+1$ so we are done. If not, then $p \equiv 1 \pmod 8$ and we have that $4 \mid h$ and so $4 \mid k$.

We now consider the case when $p \equiv 5 \pmod 8$. In this case either $p \equiv 13 \pmod{24}$ or $p \equiv 5 \pmod{24}$. We will show the existence of an eta-quotient in $M_h(\Gamma_1(p))$, since if $f(\tau) \in M_h(\Gamma_1(p))$, then $f(\tau)^{k/h} \in M_k(\Gamma_1(p))$. Suppose that $p \equiv 13 \pmod{24}$. Then $h = 6$, and using Theorems 1.2 and 1.4 one can quickly check that

$$\eta(\tau)^9\eta(p\tau)^3 \in M_6(\Gamma_1(p)).$$

Next, when $p = 5$ we have $h = 2$, and so

$$\frac{\eta(p\tau)^5}{\eta(\tau)} \in M_2(\Gamma_1(p)).$$

Finally, suppose $p \equiv 5 \pmod{24}$, with $p \neq 5$ and $k \neq 2$. It is sufficient to show that there exist eta-quotients $f_4(\tau) \in M_4(\Gamma_1(p))$ and $f_6(\tau) \in M_6(\Gamma_1(p))$ since either $4 \mid k$ or $4 \mid (k-6)$ and thus either $f_4^{k/4} \in M_k(\Gamma_1(p))$ or $f_4^{(k-6)/4} f_6 \in M_k(\Gamma_1(p))$. We check using Theorems 1.2 and 1.4 that

$$f_4(\tau) := \eta(\tau)^4\eta(p\tau)^4 \in M_4(\Gamma_1(p)),$$
$$f_6(\tau) := \eta(\tau)^9\eta(p\tau)^3 \in M_6(\Gamma_1(p)),$$

which resolves the final case. □

3. Eta-quotients of semiprime and square-free level

In this section, we generalize the results of Section 2 to semiprime level, and further to square-free levels where possible. We begin with a square-free generalization of Lemma 2.1 which extends [Arnold-Roksandich, James and Keaton 2018a, Theorem 5.8].

Theorem 3.1. *Let $k \in \mathbb{Z}$ and let $N = p_1 \cdots p_t$, a product of distinct primes with $p_i \geq 5$. Define $h = \frac{1}{2}\gcd(p_1 - 1, \ldots, p_t - 1, 24)$. There exists*

$$f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta} \in M_k^!(\Gamma_1(N))$$

if and only if $h \mid k$.

Proof. Suppose there exists

$$f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta} \in M_k^!(\Gamma_1(N)).$$

By Remark 1.8 we see that f satisfies the conditions in Theorem 1.4, and thus we can compute

$$v_{1/N} = \frac{1}{24} \sum_{\delta \mid N} \delta r_\delta. \tag{9}$$

Since $k = k_f = \frac{1}{2}\sum_{\delta \mid N} r_\delta$, (9) is equivalent to

$$2k = 24v_{1/N} - \sum_{\delta \mid N} (\delta - 1) r_\delta. \tag{10}$$

Equation (10) can be viewed as a linear Diophantine equation in the variables $v_{1/N}$ and r_δ for each $\delta \mid N$ with $\delta \neq 1$. Thus, the existence of an integer solution to (10) implies that

$$\gcd(\delta_1 - 1, \ldots, \delta_s - 1, 24) \mid 2k, \tag{11}$$

where $\delta_1, \ldots, \delta_s$ are the divisors of N. However, we note that

$$\gcd(\delta_1 - 1, \ldots, \delta_s - 1, 24) = \gcd(p_1 - 1, \ldots, p_t - 1, 24) = 2h, \tag{12}$$

which follows from the fact that $ab - 1 = (a-1)(b-1) + (a-1) + (b-1)$, so if $d \mid (a-1)$ and $d \mid (b-1)$, then $d \mid (ab-1)$. As each p_i is odd, we have that $2 \mid \gcd(p_1 - 1, \ldots, p_t - 1, 24)$. Therefore, h is an integer and we see from (11) and (12) that $h \mid k$ as desired.

Conversely, if $h \mid k$, then there exist integers $v_{1/N}$ and r_δ for each $\delta \mid N$ with $\delta \neq 1$, which give a solution to (10). Additionally defining

$$r_1 = 2k - \sum_{\delta \mid N,\, \delta \neq 1} r_\delta$$

gives that $k = \frac{1}{2}\sum_{\delta \mid N} r_\delta$. Thus from (10) we see that

$$\sum_{\delta \mid N} \delta r_\delta = 24 v_{1/N} \equiv 0 \pmod{24},$$

and so by Remark 1.3,

$$f(\tau) := \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta} \in M_k^!(\Gamma_1(N)). \qquad \square$$

The next theorem computes the sum of the orders of vanishing of an eta-quotient in $M_k^!(\Gamma_1(N))$ in terms of the divisor function

$$\sigma_a(n) := \sum_{d \mid n} d^a.$$

Theorem 3.2. *Let $N = p_1 \cdots p_t$, a product of distinct primes with $p_i \geq 5$, and let $f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta} \in M_{k_f}^!(\Gamma_1(N))$. Then*

$$\sum_{d \mid N} v_{1/d} = \frac{k\sigma_1(N)}{12}.$$

Proof. Since N is square-free, Theorem 1.4 gives that

$$\sum_{d \mid N} v_{1/d} = \sum_{d \mid N} \frac{N}{24} \sum_{\delta \mid N} \frac{\gcd(d, \delta)^2 r_\delta}{d\delta} = \frac{1}{24} \sum_{\delta \mid N} r_\delta \sum_{d \mid N} \frac{N \gcd(d, \delta)^2}{d\delta}. \tag{13}$$

Since $\sum_{\delta \mid N} r_\delta = 2k$, it suffices to show that the inner sum in (13) is $\sigma_1(N)$. We thus fix $\delta \mid N$, and aim to show that, for each divisor d' of N, there is a unique divisor $d \mid N$ such that

$$\frac{N \gcd(d, \delta)^2}{d\delta} = d'. \tag{14}$$

We first show existence. Given $d' \mid N$, set $d = \gcd(d', \delta) \gcd(N/d', N/\delta)$, which clearly divides N. Since δ, d', and d are all square-free, we observe that $\gcd(d, \delta) = \gcd(d', \delta)$, since any prime divisor of d and δ must divide $\gcd(d', \delta)$.

Our goal is to show (14), which we rewrite as

$$\frac{N}{\gcd(N/d', N/\delta)} = \frac{\delta d'}{\gcd(d', \delta)}. \tag{15}$$

The left-hand side of (15) is simply the product of the prime divisors of N which are also prime divisors of δ or d'. But this is also the right-hand side of (15) so we are done and have shown existence.

To determine uniqueness, we suppose $d_1, d_2 \mid N$ such that

$$\frac{N \gcd(d_1, \delta)^2}{d_1\delta} = \frac{N \gcd(d_2, \delta)^2}{d_2\delta},$$

namely that $d_2 \gcd(d_1, \delta)^2 = d_1 \gcd(d_2, \delta)^2$. Since N is square-free, comparing the prime factorizations of d_1, d_2, δ and considering cases yields that $d_1 = d_2$ as desired.

Since we have shown that

$$\sum_{d \mid N} \frac{N \gcd(d, \delta)^2}{d\delta} = \sigma_1(N), \tag{16}$$

we are now finished. □

Remark 3.3. When N is square-free, $\sigma_1(N) = \prod_{p \mid N}(p+1)$. Thus when $N = p_1 \cdots p_t$, a product of distinct primes with $p_i \geq 5$, and

$$f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta} \in M^!_{k_f}(\Gamma_1(N)),$$

we have by Theorem 3.2 that

$$\sum_{d \mid N} v_{1/d} = \frac{k\sigma_1(N)}{12} = \frac{k\prod_{p \mid N}(p+1)}{12}.$$

From this we can observe that $k\sigma_1(N)/12$ must always be an integer whenever k is an even integer such that $h \mid k$ for $h = \frac{1}{2}\gcd(p_1 - 1, \ldots, p_t - 1, 24)$. This is because if 3 doesn't divide $p+1$ for any $p \mid N$, then 3 must divide $p-1$ for all $p \mid N$, and so 3 divides h and thus k.

We next generalize Lemma 2.4 to square-free levels.

Lemma 3.4. *Let $N = p_1 \cdots p_t$, a product of distinct primes with $p_i \geq 5$, and define $h = \frac{1}{2}\gcd(p_1 - 1, \ldots, p_t - 1, 24)$. Suppose k is a positive integer such that $h \mid k$, $2^{t-1} \mid k$, and $4 \mid k$ if $t = 2$. Then there exists*

$$f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta} \in M_k(\Gamma_1(N)).$$

Proof. Define $f(\tau)$ by

$$f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{k/2^{t-1}}.$$

As N is square-free, there are 2^t divisors of N, and so

$$\sum_{\delta \mid N} \frac{k}{2^{t-1}} = 2k.$$

Additionally, by our hypothesis $2^t \mid (k\sigma_1(N)/12)$, so we have

$$\sum_{\delta \mid N} \delta\left(\frac{k}{2^{t-1}}\right) = \left(\frac{k}{2^{t-1}}\right)\sigma_1(N) = \frac{k\sigma_1(N)}{2^{t-1}} \equiv 0 \pmod{24}.$$

Thus, $f(\tau)$ satisfies Theorem 1.2, and so by Remark 1.8, $f(\tau) \in M^!_k(\Gamma_1(N))$. Recalling (16), we have by Theorem 1.4 that

$$v_{1/d} = \frac{k}{2^{t-1}} \cdot \frac{\sigma_1(N)}{24} = \frac{k\sigma_1(N)}{2^t \cdot 12},$$

which is a positive integer by our hypotheses since $2^t \mid \sigma_1(N)$. Thus $f(\tau) \in M_k(\Gamma_1(N))$. □

We are now ready to prove Theorem 1.10.

Proof of Theorem 1.10. First, we assume that

$$f(\tau) = \eta(\tau)^{r_1}\eta(p\tau)^{r_p}\eta(q\tau)^{r_q}\eta(N\tau)^{r_N} \in M_k(\Gamma_1(pq)),$$

where $N = pq$ is a product of distinct primes $p, q \geq 5$, and k is an even integer. By Theorem 3.1, we have that $h \mid k$. To complete this forward direction of the proof, we show that there are no eta-quotients in $M_2(\Gamma_1(pq))$ when $(p, q) \equiv (1, 5), (5, 1), (5, 5) \pmod{24}$ and $p, q > 5$. Without loss of generality, we may assume p and q are chosen so that p has a lesser or equal residue modulo 24.

Recalling Remark 2.3, we use Theorem 2.2. Fix $(p, q) \equiv (1, 5), (5, 5) \pmod{24}$ with $p, q > 5$, and suppose

$$f(\tau) = \eta(\tau)^{r_1}\eta(p\tau)^{r_p}\eta(q\tau)^{r_q}\eta(N\tau)^{r_N} \in M_2(\Gamma_1(pq)).$$

Then by Theorem 2.2,

$$|r_1| + |r_p| + |r_q| + |r_N| \leq 4\left(\frac{p+1}{p-1}\right)\left(\frac{q+1}{q-1}\right).$$

This upper bound decreases as p and q increases, so the bound will be largest when $p = 29$ and $q = 53$. This gives a bound less than 5, so, since the r_δ are integers,

$$|r_1| + |r_p| + |r_q| + |r_N| \leq 4. \tag{17}$$

Moreover, by Theorem 1.4,

$$\begin{aligned} 24v_1 &= Nr_1 + qr_p + pr_q + r_N, \\ 24v_{1/p} &= qr_1 + Nr_p + r_q + pr_N, \\ 24v_{1/q} &= pr_1 + r_p + Nr_q + qr_N, \\ 24v_{1/N} &= r_1 + pr_p + qr_q + Nr_N. \end{aligned}$$

Since $f(\tau) \in M_k(\Gamma_1(N))$, we have that $v_1, v_{1/p}, v_{1/q}, v_{1/N} \geq 0$. It follows that $r_1, r_p, r_q, r_N \geq 0$. Namely, if $r_1 < 0$, then using (17), and, assuming without loss of generality that $p < q$, we see that

$$24v_1 \leq -pq|r_1| + q|r_p| + q|r_q| + q|r_N| \leq -pq|r_1| + 4q < -pq|r_1| + pq \leq 0,$$

which contradicts that $v_1 \geq 0$. Similarly, if r_p, r_q, or r_N is negative we get contradictions for the nonnegativity of $v_{1/p}$, $v_{1/q}$, and $v_{1/N}$ respectively.

By Remark 1.8, we also know that

$$r_1 + r_p + 5r_q + 5r_N \equiv 0 \pmod{24} \tag{18}$$

if $(p, q) \equiv (1, 5) \pmod{24}$, and

$$r_1 + 5r_p + 5r_q + r_N \equiv 0 \pmod{24} \tag{19}$$

if $(p, q) \equiv (5, 5) \pmod{24}$.

However, neither (18) nor (19) have nonnegative integer solutions which satisfy (17), which is a contradiction. Thus we have shown that there are no eta-quotients in $M_2(\Gamma_1(pq))$ for $(p, q) \equiv (1, 5), (5, 1), (5, 5) \pmod{24}$ and $p, q > 5$.

We now prove the converse by construction. Namely, we need to show that if $h \mid k$ and it is not the case that $(p, q) \pmod{24} \in \{(1, 5), (5, 1), (5, 5)\}$, $p, q \neq 5$, and $k = 2$, then there does exist an eta-quotient in $M_k(\Gamma_1(pq))$.

We first note that setting $t = 2$ in Lemma 3.4 guarantees the existence of an eta-quotient in $M_k(\Gamma_1(pq))$ for distinct primes $p, q \geq 5$, when k is an even integer divisible by h such that $k(p + 1)(q + 1)/12$ is divisible by 4. Since 4 divides $k(p + 1)(q + 1)/12$ whenever 4 divides any one of $p + 1$, $q + 1$, or k, it suffices to consider only the cases of p, q, and k when $p + 1 \equiv q + 1 \equiv k \equiv 2 \pmod 4$. Consider the possible residues for (p, q) modulo 24, ordering so that the residue of p is no larger than that of q. We may immediately disregard the cases (1, 1), (1, 17), and (17, 17), since in each $4 \mid h$, and thus since $h \mid k$ they are covered by Lemma 3.4. This leaves the cases (1, 5), (1, 13), (5, 5), (5, 13), (5, 17), (13, 13), and (13, 17). It suffices to show there exists an eta-quotient in $M_h(\Gamma_1(N))$, since if $f(\tau) \in M_h(\Gamma_1(p))$, then $f(\tau)^{k/h} \in M_k(\Gamma_1(p))$. The following table gives such eta-quotients for the cases (1, 13), (5, 13), (5, 17), (13, 13), and (13, 17):

case	h	eta-quotient
(1, 13)	6	$\eta(\tau)^{11}\eta(q\tau)$
(5, 13)	2	$\eta(p\tau)\eta(q\tau)^2\eta(N\tau)$
(5, 17)	2	$\eta(p\tau)\eta(q\tau)\eta(N\tau)^2$
(13, 13)	6	$\eta(q\tau)\eta(N\tau)^{11}$
(13, 17)	2	$\eta(p\tau)^2\eta(q\tau)\eta(N\tau)$

This leaves the cases (1, 5) and (5, 5), both of which have $h = 2$. If either p or q is 5, then $M_2(\Gamma(5)) \subset M_2(\Gamma(N))$, and so by Theorem 1.9 there will exist an eta-quotient in $M_2(\Gamma(N))$. We thus assume that neither p nor q is equal to 5. As every even integer $k \geq 4$ can be written as a linear combination of 4 and 6, it suffices to show the existence of an eta-quotient in both $M_4(\Gamma_1(N))$ and $M_6(\Gamma_1(N))$. By Lemma 3.4, we have the existence of an eta-quotient in $M_4(\Gamma_1(N))$. Moreover, one can check using Theorem 1.2 that $\eta(\tau)^3\eta(N\tau)^9 \in M_6(\Gamma_1(N))$ when $(p, q) \equiv (1, 5) \pmod{24}$, and $\eta(q\tau)^3\eta(N\tau)^9 \in M_6(\Gamma_1(N))$ when $(p, q) \equiv (5, 5) \pmod{24}$. □

4. Elliptic curves and eta-Quotients

In this section, we prove Theorems 1.11 and 1.12, and conclude by describing the method we used to find these examples.

Proof of Theorem 1.11. First using dimension formulas (Theorems 3.5.1 and 3.1.1 in [Diamond and Shurman 2005] for example), we calculate that $\dim_{\mathbb{C}} S_2(\Gamma_0(35)) = 3$.

By Theorems 1.2 and 1.4, we see that the following three eta-quotients are members of $S_2(\Gamma_0(35))$:

$$g_1(\tau) := \eta(\tau)\eta(5\tau)\eta(7\tau)\eta(35\tau),$$
$$g_2(\tau) := \eta(\tau)^2\eta(35\tau)^2,$$
$$g_3(\tau) := \eta(5\tau)^2\eta(7\tau)^2.$$

The q-expansions of g_1, g_2, g_3 begin with

$$g_1(\tau) = q^2 - q^3 - q^4 + q^8 + q^9 + O(q^{10}),$$
$$g_2(\tau) = q^3 - 2q^4 - q^5 + 2q^6 + q^7 + 2q^8 - 2q^9 + O(q^{10}),$$
$$g_3(\tau) = q - 2q^6 - 2q^8 + O(q^{10}).$$

Since each q-expansion starts with a different power of q, we can quickly determine that g_1, g_2, g_3 are linearly independent. Thus, they form a basis of $S_2(\Gamma_0(35))$, and by Theorem 1.1, any elliptic curve of conductor 35 must be a linear combination of g_1, g_2, and g_3. However, one can see for example from the L-functions and modular forms database [LMFDB 2019] that there is only one isogeny class of elliptic curves of conductor 35 [LMFDB 2019, elliptic curve isogeny class 35.a], and thus only one attached modular form, $f(\tau) \in S_2(\Gamma_0(35))$ [LMFDB 2019, modular form 35.2.1.a]. The q-expansion of f begins with

$$f(\tau) = q + q^3 - 2q^4 - q^5 + q^7 - 2q^9 + O(q^{10}),$$

and since the bound in Theorem 1.5 is 8 in this case, we see by Remark 1.6 that $f(\tau) = g_2(\tau) + g_3(\tau)$, as desired. □

We now turn to the proof of Theorem 1.12, which requires more finesse.

Proof of Theorem 1.12. As with conductor 35, there is only one isogeny class of elliptic curves of conductor 55 [LMFDB 2019, elliptic curve isogeny class 55.a], and thus only one attached modular form, $f(\tau) \in S_2(\Gamma_0(55))$ [LMFDB 2019, modular form 55.2.1.a]. The q-expansion of f begins with

$$f(\tau) = q + q^2 - q^4 + q^5 - 3q^8 - 3q^9 + q^{10} - q^{11} + 2q^{13} + O(q^{15}).$$

The bound in Theorem 1.5 is 12 in this case, so we see by Remark 1.6 that modular forms in $S_2(\Gamma_0(55))$ are determined by their Fourier coefficients up to q^{13}.

Using dimension formulas, we calculate that $\dim_{\mathbb{C}} S_2(\Gamma_0(55)) = 5$. However, there are only three linearly independent eta-quotients in $S_2(\Gamma_0(55))$ given by

$$g_1(\tau) := \eta(\tau)\eta(5\tau)\eta(11\tau)\eta(55\tau),$$
$$g_2(\tau) := \eta(\tau)^2\eta(11\tau)^2,$$
$$g_3(\tau) := \eta(5\tau)^2\eta(55\tau)^2,$$

with q-expansions beginning with

$$g_1(\tau) = q^3 - q^4 - q^5 + q^9 + 2q^{10} - 2q^{13} + O(q^{15}),$$
$$g_2(\tau) = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} + 4q^{13} + 4q^{14} + O(q^{15}),$$
$$g_3(\tau) = q^5 - 2q^{10} + O(q^{15}).$$

We thus use a method originating in [Rouse and Webb 2015], which is to multiply $f(\tau)$ by an eta-quotient $a(\tau)$ in order to push the product $f(\tau)a(\tau)$ into a higher weight space that is generated by eta-quotients. Then f can be written as a linear combination of weakly holomorphic eta-quotients in $M_2^!(\Gamma_0(55))$.

Consider the eta-quotient

$$a(\tau) = \eta(\tau)^3\eta(5\tau)^3\eta(11\tau)^3\eta(55\tau)^3.$$

We see that $a(\tau) \in S_6(\Gamma_0(55))$ by Theorems 1.2 and 1.4, and so $a(\tau)f(\tau) \in S_8(\Gamma_0(55))$. Since the bound from Theorem 1.5 is 48 in this case, modular forms in $S_8(\Gamma_0(55))$ are determined by their Fourier coefficients up to q^{49}. We see that the q-expansion for $a(\tau)$ begins with

$$\begin{aligned} a(\tau) = q^9 &- 3q^{10} + 5q^{12} - 3q^{14} + 2q^{15} - 15q^{17} + 9q^{19} + 18q^{20} + 9q^{21} - 15q^{23} \\ &- 33q^{24} - 6q^{25} - 6q^{26} + 25q^{27} + 45q^{28} + 33q^{29} - 49q^{30} - 63q^{31} + 45q^{34} \\ &+ 60q^{35} + 45q^{36} - 15q^{37} - 75q^{38} - 62q^{39} - 78q^{40} + 66q^{41} + 15q^{42} \\ &- 15q^{43} + 21q^{45} + 117q^{46} - 15q^{47} + 55q^{48} - 63q^{49} + O(q^{50}), \end{aligned}$$

whereas the expansion for $f(\tau)$ begins with

$$\begin{aligned} f(\tau) = q &+ q^2 - q^4 + q^5 - 3q^8 - 3q^9 + q^{10} - q^{11} + 2q^{13} - q^{16} + 6q^{17} \\ &- 3q^{18} - 4q^{19} - q^{20} - q^{22} + 4q^{23} + q^{25} + 2q^{26} + 6q^{29} - 8q^{31} \\ &+ 5q^{32} + 6q^{34} + 3q^{36} - 2q^{37} - 4q^{38} - 3q^{40} + 2q^{41} + 4q^{43} \\ &+ q^{44} - 3q^{45} + 4q^{46} - 12q^{47} - 7q^{49} + O(q^{50}). \end{aligned}$$

Thus we see that the q-expansion for the product $f(\tau)a(\tau)$ starts with

$$\begin{aligned} f(\tau)a(\tau) = q^{10} &- 2q^{11} - 3q^{12} + 4q^{13} + 9q^{14} - 6q^{15} - 6q^{16} + 4q^{17} - 6q^{18} \\ &- 10q^{19} - 8q^{20} + 30q^{21} + 28q^{22} - 8q^{23} - 33q^{24} + 20q^{25} \\ &+ 22q^{26} - 66q^{27} - 25q^{28} + 12q^{29} + 21q^{30} - 52q^{31} + 44q^{32} \\ &+ 96q^{33} + 19q^{34} + 82q^{35} - 105q^{36} - 86q^{37} + 29q^{38} + 108q^{39} \\ &- 148q^{40} - 106q^{41} + 159q^{42} - 260q^{43} - 136q^{44} + 36q^{45} \\ &+ 261q^{46} - 22q^{47} + 309q^{48} + 430q^{49} + O(q^{50}). \end{aligned}$$

i	$g_i(\tau)$	i	$g_i(\tau)$
1	$\eta(\tau)^4\eta(5\tau)^4\eta(11\tau)^4\eta(55\tau)^4$	21	$\eta(\tau)^5\eta(5\tau)^{-1}\eta(11\tau)^3\eta(55\tau)^9$
2	$\eta(\tau)^5\eta(5\tau)^3\eta(11\tau)^5\eta(55\tau)^3$	22	$\eta(\tau)^9\eta(5\tau)^3\eta(11\tau)^5\eta(55\tau)^{-1}$
3	$\eta(\tau)^3\eta(5\tau)^5\eta(11\tau)^3\eta(55\tau)^5$	23	$\eta(\tau)^{-1}\eta(5\tau)^5\eta(11\tau)^3\eta(55\tau)^9$
4	$\eta(\tau)^2\eta(5\tau)^6\eta(11\tau)^2\eta(55\tau)^6$	24	$\eta(\tau)^4\eta(11\tau)^2\eta(55\tau)^{10}$
5	$\eta(\tau)^6\eta(5\tau)^2\eta(11\tau)^6\eta(55\tau)^2$	25	$\eta(\tau)^3\eta(5\tau)\eta(11\tau)\eta(55\tau)^{11}$
6	$\eta(\tau)^7\eta(5\tau)\eta(11\tau)\eta(55\tau)^7$	26	$\eta(\tau)^2\eta(5\tau)^2\eta(55\tau)^{12}$
7	$\eta(\tau)^6\eta(5\tau)^6\eta(11\tau)^2\eta(55\tau)^2$	27	$\eta(\tau)\eta(5\tau)^3\eta(11\tau)^{-1}\eta(55\tau)^{13}$
8	$\eta(\tau)^7\eta(5\tau)\eta(11\tau)^7\eta(55\tau)$	28	$\eta(\tau)^6\eta(5\tau)^{-2}\eta(11\tau)^{-2}\eta(55\tau)^{14}$
9	$\eta(\tau)\eta(5\tau)^7\eta(11\tau)\eta(55\tau)^7$	29	$\eta(5\tau)^4\eta(11\tau)^{-2}\eta(55\tau)^{14}$
10	$\eta(\tau)^8\eta(11\tau)^2\eta(55\tau)^6$	30	$\eta(11\tau)^2\eta(55\tau)^{14}$
11	$\eta(\tau)^7\eta(5\tau)^5\eta(11\tau)^3\eta(55\tau)$	31	$\eta(\tau)\eta(5\tau)^{-1}\eta(11\tau)^{-3}\eta(55\tau)^{19}$
12	$\eta(\tau)\eta(5\tau)^3\eta(11\tau)^5\eta(55\tau)^7$	32	$\eta(\tau)\eta(5\tau)^7\eta(11\tau)^7\eta(55\tau)$
13	$\eta(\tau)^5\eta(5\tau)^7\eta(11\tau)\eta(55\tau)^3$	33	$\eta(\tau)^2\eta(5\tau)^2\eta(11\tau)^6\eta(55\tau)^6$
14	$\eta(5\tau)^8\eta(55\tau)^8$	34	$\eta(\tau)^3\eta(5\tau)\eta(11\tau)^7\eta(55\tau)^5$
15	$\eta(\tau)^8\eta(11\tau)^8$	35	$\eta(\tau)^2\eta(5\tau)^6\eta(11\tau)^8$
16	$\eta(\tau)^9\eta(5\tau)^{-1}\eta(11\tau)^3\eta(55\tau)^5$	36	$\eta(\tau)^6\eta(5\tau)^2\eta(55\tau)^8$
17	$\eta(\tau)^5\eta(5\tau)^3\eta(11\tau)^{-1}\eta(55\tau)^9$	37	$\eta(5\tau)^8\eta(11\tau)^6\eta(55\tau)^2$
18	$\eta(\tau)^{-1}\eta(5\tau)^9\eta(11\tau)^5\eta(55\tau)^3$	38	$\eta(\tau)^4\eta(11\tau)^8\eta(55\tau)^4$
19	$\eta(\tau)^3\eta(5\tau)^5\eta(11\tau)^9\eta(55\tau)^{-1}$	39	$\eta(\tau)^8\eta(5\tau)^4\eta(11\tau)^4$
20	$\eta(\tau)^3\eta(5\tau)^9\eta(11\tau)^5\eta(55\tau)^{-1}$	40	$\eta(5\tau)^4\eta(11\tau)^4\eta(55\tau)^8$

Table 1. Eta-quotient basis of $S_8(\Gamma_0(55))$.

Moreover, we calculate that $\dim_{\mathbb{C}} S_8(\Gamma_0(55)) = 40$, and compute a basis of $S_8(\Gamma_0(55))$ consisting only of eta-quotients $\{g_1, \ldots, g_{40}\}$, which are given in Table 1.

From their q-expansions, we calculate that

$$f(\tau)a(\tau) = \sum_{i=1}^{40} c_i g_i(\tau),$$

where the coefficients c_i are given in Table 2.

Thus, we can write

$$f(\tau) = \sum_{i=1}^{40} c_i \frac{g_i(\tau)}{a(\tau)}, \tag{20}$$

where the simplified eta-quotients $g_i(\tau)/a(\tau)$ are given in Table 3. □

We conclude by describing in an algorithmic fashion the method we used to obtain Theorems 1.11 and 1.12. We have seen that both of these theorems, once

i	c_i
1	$-\frac{6008649555929309389497}{819506238451459924562}$
2	$-\frac{502520890503551696366}{40975311922572996281}$
3	$-\frac{1807946684661764776357 4}{12292593576771898868 43}$
4	$-\frac{9707713817545985330315}{1639012476902919849124}$
5	$-\frac{256094683592582994017}{819506238451459924562}$
6	$-\frac{80175532732356749 5694}{10782976821729735 8495}$
7	$-\frac{5830988018825221370539}{12292593576771898868430}$
8	$-\frac{779344877836568799883}{24585187153543797736 86}$
9	$\frac{8222210796731963837135}{122925935767718988 6843}$
10	$-\frac{1103662021658033427 3019}{24585187153543797736860}$
11	$-\frac{1229041712172689367604}{6146296788385949434215}$
12	$-\frac{1115615505140996841 40912}{2048765596128649811405}$
13	$\frac{1071144133820889620 36}{2048765596128649811405}$
14	$-\frac{341358820591409973660}{40975311922572996281}$
15	$-\frac{152612595137164539575}{24585187153543797736 86}$
16	0
17	$-\frac{7213279306787331549849}{819506238451459924562}$
18	$\frac{32952053389588166394}{40975311922572996281}$
19	$\frac{30522519027432907915}{24585187153543797736 86}$
20	0
21	$-\frac{1552252178781785948225}{819506238451459924562}$
22	0
23	$\frac{15742588981996697524}{11074408627722431413}$
24	$-\frac{1162843533187880300 11856}{12292593576771898868 43}$
25	$-\frac{5325608941045211094 82105}{1639012476902919849124}$
26	$-\frac{1514700568262560995033025}{24585187153543797736 86}$
27	$-\frac{5615863383025394295 00115}{819506238451459924562}$
28	$-\frac{1498106192582562197 80}{40975311922572996281}$
29	$-\frac{1707665728361679867 62205}{819506238451459924562}$
30	$-\frac{8163397022278994275 2773}{40975311922572996281}$
31	$\frac{1515683437720980945 48465}{819506238451459924562}$
32	$\frac{8450175486905929205 02}{6146296788385949434215}$
33	$-\frac{3430455824898738018 97249}{12292593576771898868430}$
34	$-\frac{7502341546089874832 40971}{24585187153543797736860}$
35	$-\frac{2970842391989560281 46}{6146296788385949434215}$
36	$-\frac{3083465007459258138 7973}{20487655961286498114 05}$
37	$-\frac{9228870808743396499}{10782976821729735 8495}$
38	$-\frac{4131906244671465777 242}{40975311922572996281}$
39	$-\frac{30522519027432907915}{24585187153543797736 86}$
40	$-\frac{1314644674905464625 1719}{819506238451459924562}$

Table 2. Coefficients c_i.

i	$g_i(\tau)/a(\tau)$	i	$g_i(\tau)/a(\tau)$
1	$\eta(\tau)\eta(5\tau)\eta(11\tau)\eta(55\tau)$	21	$\eta(\tau)^2\eta(5\tau)^{-4}\eta(55\tau)^6$
2	$\eta(\tau)^2\eta(11\tau)^2$	22	$\eta(\tau)^6\eta(11\tau)^2\eta(55\tau)^{-4}$
3	$\eta(5\tau)^2\eta(55\tau)^2$	23	$\eta(\tau)^{-4}\eta(5\tau)^2\eta(55\tau)^6$
4	$\eta(\tau)^{-1}\eta(5\tau)^3\eta(11\tau)^{-1}\eta(55\tau)^3$	24	$\eta(\tau)\eta(5\tau)^{-3}\eta(11\tau)^{-1}\eta(55\tau)^7$
5	$\eta(\tau)^3\eta(5\tau)^{-1}\eta(11\tau)^3\eta(55\tau)^{-1}$	25	$\eta(5\tau)^{-2}\eta(11\tau)^{-2}\eta(55\tau)^8$
6	$\eta(\tau)^4\eta(5\tau)^{-2}\eta(11\tau)^{-2}\eta(55\tau)^4$	26	$\eta(\tau)^{-1}\eta(5\tau)^{-1}\eta(11\tau)^{-3}\eta(55\tau)^9$
7	$\eta(\tau)^3\eta(5\tau)^3\eta(11\tau)^{-1}\eta(55\tau)^{-1}$	27	$\eta(\tau)^{-2}\eta(11\tau)^{-4}\eta(55\tau)^{10}$
8	$\eta(\tau)^4\eta(5\tau)^{-2}\eta(11\tau)^4\eta(55\tau)^{-2}$	28	$\eta(\tau)^3\eta(5\tau)^{-5}\eta(11\tau)^{-5}\eta(55\tau)^{11}$
9	$\eta(\tau)^{-2}\eta(5\tau)^4\eta(11\tau)^{-2}\eta(55\tau)^4$	29	$\eta(\tau)^{-3}\eta(5\tau)\eta(11\tau)^{-5}\eta(55\tau)^{11}$
10	$\eta(\tau)^5\eta(5\tau)^{-3}\eta(11\tau)^{-1}\eta(55\tau)^3$	30	$\eta(\tau)^{-3}\eta(5\tau)^{-3}\eta(11\tau)^{-1}\eta(55\tau)^{11}$
11	$\eta(\tau)^4\eta(5\tau)^2\eta(55\tau)^{-2}$	31	$\eta(\tau)^{-2}\eta(5\tau)^{-4}\eta(11\tau)^{-6}\eta(55\tau)^{16}$
12	$\eta(\tau)^{-2}\eta(11\tau)^2\eta(55\tau)^4$	32	$\eta(\tau)^{-2}\eta(5\tau)^4\eta(11\tau)^4\eta(55\tau)^{-2}$
13	$\eta(\tau)^2\eta(5\tau)^4\eta(11\tau)^{-2}$	33	$\eta(\tau)^{-1}\eta(5\tau)^{-1}\eta(11\tau)^3\eta(55\tau)^3$
14	$\eta(\tau)^{-3}\eta(5\tau)^5\eta(11\tau)^{-3}\eta(55\tau)^5$	34	$\eta(5\tau)^{-2}\eta(11\tau)^4\eta(55\tau)^2$
15	$\eta(\tau)^5\eta(5\tau)^{-3}\eta(11\tau)^5\eta(55\tau)^{-3}$	35	$\eta(\tau)^{-1}\eta(5\tau)^3\eta(11\tau)^5\eta(55\tau)^{-3}$
16	$\eta(\tau)^6\eta(5\tau)^{-4}\eta(55\tau)^2$	36	$\eta(\tau)^3\eta(5\tau)^{-1}\eta(11\tau)^{-3}\eta(55\tau)^5$
17	$\eta(\tau)^2\eta(11\tau)^{-4}\eta(55\tau)^6$	37	$\eta(\tau)^{-3}\eta(5\tau)^5\eta(11\tau)^3\eta(55\tau)^{-1}$
18	$\eta(\tau)^{-4}\eta(5\tau)^6\eta(11\tau)^2$	38	$\eta(\tau)\eta(5\tau)^{-3}\eta(11\tau)^5\eta(55\tau)$
19	$\eta(5\tau)^2\eta(11\tau)^6\eta(55\tau)^{-4}$	39	$\eta(\tau)^5\eta(5\tau)\eta(11\tau)\eta(55\tau)^{-3}$
20	$\eta(5\tau)^6\eta(11\tau)^2\eta(55\tau)^{-4}$	40	$\eta(\tau)^{-3}\eta(5\tau)\eta(11\tau)\eta(55\tau)^5$

Table 3. Eta-quotients $g_i(\tau)/a(\tau)$.

discovered, can be proved in a straightforward manner. However, how to find results like these may not be immediately apparent. Our approach, which is inspired by [Pathakjee, RosnBrick and Yoong 2012], utilizes results from Section 3 and has the potential to generate many new examples. We note that many of the steps require the aid of mathematical software to be practical; we used SageMath [2018]. We do not know whether the process we outline will necessarily terminate.

Step 1: Fix a semiprime $N = pq$ satisfying the conditions in Theorem 1.10 with $k = 2$ that is the conductor of an elliptic curve E. By the modularity theorem (Theorem 1.1) we know that E has an associated modular form $f(\tau) \in S_2(\Gamma_0(N))$.

Step 2: Compute $d_N = \dim_{\mathbb{C}} S_2(\Gamma_0(N))$.

Step 3: Compute all partitions of $(p+1)(q+1)/6$ into exactly four parts, and construct distinct rearrangements in order to get a complete list of all possible tuples $(v_1, v_{1/p}, v_{1/q}, v_N) \in \mathbb{N}^4$ satisfying

$$v_1 + v_{1/p} + v_{1/q} + v_{1/N} = \tfrac{1}{6}(p+1)(q+1). \tag{21}$$

By Theorem 3.2, we know that any eta-quotient in $S_2(\Gamma_0(N))$ must have orders of vanishing $v_1, v_{1/p}, v_{1/q}, v_{1/N} \geq 1$ that satisfy (21).

Step 4: For each tuple $(v_1, v_{1/p}, v_{1/q}, v_N)$ obtained in Step 3, use Theorem 1.4 to construct the system of four equations in the four unknowns (r_1, r_p, r_q, r_N)

$$\begin{aligned} 24v_1 &= Nr_1 + qr_p + pr_q + r_N, \\ 24v_{1/p} &= qr_1 + Nr_p + r_q + pr_N, \\ 24v_{1/q} &= pr_1 + r_p + Nr_q + qr_N, \\ 24v_{1/N} &= r_1 + pr_p + qr_q + Nr_N, \end{aligned}$$

and solve for the unique solution $(r_1, r_p, r_q, r_N) \in \mathbb{Q}^4$.

Step 5: For each tuple (r_1, r_p, r_q, r_N) from Step 4 that has integer entries, let

$$g(\tau) = \eta(\tau)^{r_1}\eta(p\tau)^{r_p}\eta(q\tau)^{r_q}\eta(N\tau)^{r_N},$$

and use Theorems 1.2 and 1.4 to check whether $g(\tau) \in S_2(\Gamma_0(N))$. List all such $g \in S_2(\Gamma_0(N))$.

Step 6: Construct a maximally sized linearly independent set of eta-quotients from the list in Step 5, using linear algebra.

Step 7: If the set from Step 6 has size d_N, then it forms a basis of $S_2(\Gamma_0(N))$. In this case, compute the Sturm bound from Theorem 1.5 and write f as a linear combination of the basis from Step 6. If not, go to Step 8.

Step 8: Repeat Steps 2–6 for weights $2, 4, 6, \ldots$ until a weight k is found such that $S_k(\Gamma_0(N))$ has a basis of eta-quotients, and $S_{k-2}(\Gamma_0(N))$ contains an eta-quotient $a(\tau)$. Compute the Sturm bound from Theorem 1.5 and write $f(\tau)a(\tau)$ as a linear combination of the basis. Divide through by $a(\tau)$ to write $f(\tau)$ as a linear combination of eta-quotients in $M_2^!(\Gamma_0(N))$.

References

[Ahlgren and Ono 2001] S. Ahlgren and K. Ono, "Congruence properties for the partition function", *Proc. Natl. Acad. Sci. USA* **98**:23 (2001), 12882–12884. MR Zbl

[Arnold-Roksandich, James and Keaton 2018a] A. Arnold-Roksandich, K. James, and R. Keaton, "Counting eta-quotients of prime level", *Involve* **11**:5 (2018), 827–844. MR Zbl

[Arnold-Roksandich, James and Keaton 2018b] A. Arnold-Roksandich, K. James, and R. Keaton, "Proceedings paper for REU project involving counting eta-quotients", preprint, 2018. arXiv

[Biagioli 1990] A. J. F. Biagioli, "The construction of modular forms as products of transforms of the Dedekind eta function", *Acta Arith.* **54**:4 (1990), 273–300. MR Zbl

[Borcherds 1992] R. E. Borcherds, "Monstrous moonshine and monstrous Lie superalgebras", *Invent. Math.* **109**:2 (1992), 405–444. MR Zbl

[Conway and Norton 1979] J. H. Conway and S. P. Norton, "Monstrous moonshine", *Bull. London Math. Soc.* **11**:3 (1979), 308–339. MR Zbl

[Diamond and Shurman 2005] F. Diamond and J. Shurman, *A first course in modular forms*, Graduate Texts in Mathematics **228**, Springer, 2005. MR Zbl

[Dummit, Kisilevsky and McKay 1985] D. Dummit, H. Kisilevsky, and J. McKay, "Multiplicative products of η-functions", pp. 89–98 in *Finite groups*: *coming of age* (Montreal, 1982), edited by J. McKay, Contemp. Math. **45**, Amer. Math. Soc., Providence, RI, 1985. MR Zbl

[Gordon and Hughes 1993] B. Gordon and K. Hughes, "Multiplicative properties of η-products, II", pp. 415–430 in *A tribute to Emil Grosswald*: *number theory and related analysis*, edited by M. Knopp and M. Sheingorn, Contemp. Math. **143**, Amer. Math. Soc., Providence, RI, 1993. MR Zbl

[Kilford 2007] L. J. P. Kilford, "Generating spaces of modular forms with η-quotients", *JP J. Algebra Number Theory Appl.* **8**:2 (2007), 213–226. MR Zbl

[Lemke Oliver 2013] R. J. Lemke Oliver, "Eta-quotients and theta functions", *Adv. Math.* **241** (2013), 1–17. MR Zbl

[Ligozat 1974] G. Ligozat, *Courbes modulaires de genre* 1, Ph.D. thesis, Université Paris XI, 1974, available at http://sites.mathdoc.fr/PMO/PDF/L_LIGOZAT-78.pdf. MR

[LMFDB 2019] The LMFDB Collaboration, "The L-functions and modular forms database", electronic reference, 2019, available at http://www.lmfdb.org. Online; accessed 3 August 2018.

[Martin 1996] Y. Martin, "Multiplicative η-quotients", *Trans. Amer. Math. Soc.* **348**:12 (1996), 4825–4856. MR Zbl

[Martin and Ono 1997] Y. Martin and K. Ono, "Eta-quotients and elliptic curves", *Proc. Amer. Math. Soc.* **125**:11 (1997), 3169–3176. MR Zbl

[Murty, Dewar and Graves 2015] M. R. Murty, M. Dewar, and H. Graves, *Problems in the theory of modular forms*, Institute of Mathematical Sciences Lecture Notes **1**, Hindustan Book Agency, New Delhi, 2015. MR Zbl

[Newman 1957] M. Newman, "Construction and application of a class of modular functions", *Proc. London Math. Soc.* (3) **7** (1957), 334–350. MR Zbl

[Newman 1959] M. Newman, "Construction and application of a class of modular functions, II", *Proc. London Math. Soc.* (3) **9** (1959), 373–387. MR Zbl

[Ono 2000] K. Ono, "Distribution of the partition function modulo m", *Ann. of Math.* (2) **151**:1 (2000), 293–307. MR Zbl

[Ono 2004] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics **102**, Amer. Math. Soc., Providence, RI, 2004. MR Zbl

[Pathakjee, RosnBrick and Yoong 2012] D. Pathakjee, Z. RosnBrick, and E. Yoong, "Elliptic curves, eta-quotients and hypergeometric functions", *Involve* **5**:1 (2012), 1–8. MR Zbl

[Rouse and Webb 2015] J. Rouse and J. J. Webb, "On spaces of modular forms spanned by eta-quotients", *Adv. Math.* **272** (2015), 200–224. MR Zbl

[SageMath 2018] The Sage Developers, *SageMath, the Sage Mathematics Software System*, 2018, available at http://www.sagemath.org. Version 8.3.

[Taylor and Wiles 1995] R. Taylor and A. Wiles, "Ring-theoretic properties of certain Hecke algebras", *Ann. of Math.* (2) **141**:3 (1995), 553–572. MR Zbl

[Wiles 1995] A. Wiles, "Modular elliptic curves and Fermat's last theorem", *Ann. of Math.* (2) **141**:3 (1995), 443–551. MR Zbl

Received: 2020-08-04 Accepted: 2020-08-11

allenm3@oregonstate.edu *Oregon State University, Corvallis, OR, United States*
n.anderson@se19.qmul.ac.uk *Queen Mary University of London, London, United Kingdom*
asimina.hamakiotes@uconn.edu *University of Connecticut, Storrs, CT, United States*
benjamin.oltsik@uconn.edu *University of Connecticut, Storrs, CT, United States*
swisherh@math.oregonstate.edu *Department of Mathematics, Oregon State University, Corvallis, OR, United States*

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2020 is US $205/year for the electronic version, and $275/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2020 vol. 13 no. 5

1944-4176(2020)13:5;1-P