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a journal of mathematics

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(Communicated by Filip Saidak)

The sieve of Eratosthenes (SoE) is a well-known method of extracting the set of prime numbers \mathbb{P} from the set positive integers \mathbb{N} . Applying the SoE again to the index of the prime numbers will result in the set of prime-indexed primes $\mathbb{P}_2 = \{3, 5, 11, 17, 31, \dots\}$. More generally, the application of the SoE k -times will yield the set \mathbb{P}_k of k -th order primes. In this paper, we give an upper bound for the n -th k -order prime as well as some results relating to number-theoretic functions over \mathbb{P}_k .

1. Introduction

This paper lies within the intersection of two topics in analytic number theory: abstract analytic number theory and the prime-indexed primes. In this section we give brief summaries of these topics followed by our results, whose proofs are presented in the subsequent sections.

1.1. Abstract analytic number theory. We begin with a set of *generalized primes* $\mathcal{P} := \{p_i \in \mathbb{R} \mid 1 < p_1 \leq p_2 \leq \dots < \infty\}$ out of which the set of *generalized integers* $\mathcal{N} := \{n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \mid p_i \in \mathcal{P}, \alpha_i \in \mathbb{N}\}$ is constructed; that is, \mathcal{N} consists of all possible finite words over \mathcal{P} . One of the main goals of abstract analytic number theory is to describe the asymptotic behavior of, and the relationship between, the two counting functions

$$\pi_{\mathcal{P}}(x) := \sum_{\substack{p < x \\ p \in \mathcal{P}}} 1 \quad \text{and} \quad N_{\mathcal{P}}(x) := \sum_{\substack{n < x \\ n \in \mathcal{N}}} 1. \quad (1)$$

For example, what growth conditions must we impose on $N_{\mathcal{P}}(x)$ so that $\pi_{\mathcal{P}}(x)$ satisfies the classical prime number theorem (PNT) $\pi_{\mathcal{P}}(x) \sim x / \log x$? One might naturally suppose that $N_{\mathcal{P}}(x) \sim x$ may be enough to guarantee the PNT, but Beurling [1937] showed this is not the case. In particular, he proved that if $N_{\mathcal{P}}(x) = Ax + O(x / \log^{\gamma} x)$, $A > 0$, and $\gamma > \frac{3}{2}$, then the PNT follows; however, he also showed

MSC2010: 11A41, 11N37, 11N80.

Keywords: prime-indexed primes, abstract analytic number theory, Beurling zeta function.

the PNT can fail if $\gamma = \frac{3}{2}$. For more on the PNT and Beurling's generalization of it, we refer the reader to Sections 1.1, 6.2, and 8.4 of [Montgomery and Vaughan 2007].

Central to the proof of Beurling's generalized PNT, and arguably the most fundamental object in abstract analytic number theory, is the *Beurling zeta function* $\zeta_{\mathcal{P}}(s)$, which is defined as

$$\zeta_{\mathcal{P}}(s) := \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n \in \mathcal{N}} \frac{1}{n^s} \quad (2)$$

for $s \in \mathbb{C}$ wherever it converges.¹ Classically, many important arithmetic functions arise as the coefficients of Dirichlet series involving the Riemann zeta function $\zeta(s) := \zeta_{\mathbb{P}}(s)$; see Chapter 1 of [Titchmarsh 1986]. The von Mangoldt function $\Lambda(n)$, for instance, satisfies

$$\frac{\zeta'(s)}{\zeta(s)} = - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \text{where } \Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Replacing the Riemann zeta function with the Beurling zeta function, we arrive at the generalized von Mangoldt function $\Lambda_{\mathcal{P}}(n)$ defined over \mathcal{N} ; that is,

$$\frac{\zeta'_{\mathcal{P}}(s)}{\zeta_{\mathcal{P}}(s)} = - \sum_{n \in \mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^s}, \quad \text{where } \Lambda_{\mathcal{P}}(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for } p \in \mathcal{P}, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper we will restrict our attention to the generalized prime counting function $\pi_{\mathcal{P}}(n)$, the n -th prime function $\pi_{\mathcal{P}}^{-1}(n)$, the von Mangoldt function $\Lambda_{\mathcal{P}}(n)$, and the Chebyshev theta function

$$\theta_{\mathcal{P}}(n) := \sum_{\substack{p < x \\ p \in \mathcal{P}}} \log p.$$

For further information on the Beurling zeta function and the various arithmetic functions associated to it, see the excellent summary given in Chapter 1 of [Diamond and Zhang 2016], as well as the more detailed Chapter 2 of [Knopfmacher 1990].

1.2. The k -th order primes. The sieve of Eratosthenes (SoE) is a well-known method of extracting the set of prime numbers \mathbb{P} from the set of positive integers \mathbb{N} . Applying the SoE again to the index of the prime numbers will result in the set of prime-indexed primes $\mathbb{P}_2 = \{3, 5, 11, 17, 31, 41, 59, 67, 83, \dots\}$. More generally, the application of the SoE k -times will yield the set \mathbb{P}_k of k -th order primes. Regarding \mathbb{P}_k as a set of generalized primes, one also has the corresponding set \mathbb{N}_k

¹It is known (see Proposition 2.1 in Chapter 4 of [Knopfmacher 1990]) that the Euler product and the Dirichlet series converge in the same half-plane and are equal.

of generalized integers (as constructed above in Section 1.1). It follows by definition that the corresponding prime counting functions are given by

$$\pi_k(x) := \pi_{\mathbb{P}_k}(x) = \underbrace{(\pi \circ \cdots \circ \pi)}_{k\text{-times}}(x),$$

and the PNT implies $\pi_k(x) \sim x / \log^k x$ (see Lemma 8 below). Furthermore, if we set $\pi^{-1}(n) := p_n$, where p_n is the n -th prime, then the n -th k -order prime is given by

$$\pi_{-k}(n) := \pi_{\mathbb{P}_k}^{-1}(n) = \underbrace{(\pi^{-1} \circ \cdots \circ \pi^{-1})}_{k\text{-times}}(n).$$

The k -th order primes \mathbb{P}_k and corresponding functions $\pi_{\pm k}(n)$ have been studied recently in [Broughan and Barnett 2009; Bayless et al. 2013], with emphasis on the $k = 2$, prime-indexed prime case. In addition to generalizing fundamental questions like the Goldbach and twin prime conjectures to \mathbb{P}_2 , bounds have been derived for $\pi_{\pm 2}(n)$, as well as $\pi_k(n)$ for $k \geq 2$. For instance, Broughan and Barnett give the bounds

$$\begin{aligned} \pi_{-2}(n) &< n(\log n + \log \log n)(\log n + 2 \log \log n) - n \log n + O(n \log \log n), \\ \pi_{-2}(n) &> n(\log n + \log \log n)(\log n + 2 \log \log n) - 3n \log n + O(n \log \log n). \end{aligned}$$

The goal of this paper is to continue this line of inquiry against the backdrop of abstract analytic number theory.

1.3. Results. Our main result is a bound for $\pi_{-k}(n)$ for all $k \geq 1$. Consequently, the following theorem, together with the PNT, bound $\pi_k(n)$ for all $k \in \mathbb{Z}$.

Theorem 1. *For any integer $k \geq 1$ the function $\pi_{-k}(n)$ satisfies*

$$\pi_{-k}(n) = n \prod_{j=1}^k f(n, j) + O_k(n \log^{k-2} n \log \log n), \tag{3}$$

where $f(n, j) := \log n + j \log \log n - 1$.

The proof of Theorem 1 is given below in Section 2. Ideally, one would like to determine the precise relation between k and the error constants in the theorem.

The next two theorems are generalizations of Theorems 9.1 and 9.2 in [De Koninck and Ivić 1980] involving sums of arithmetic functions. The first involves the k -th order prime counting function $\pi_k(n)$.

Theorem 2. *For any integer $k \geq 1$ the function $\pi_k(x)$ satisfies*

$$\sum'_{n \leq x} \frac{1}{\pi_k(n)} = \frac{\log^{k+1} x}{k+1} + O_k(\log^k x \log \log x), \tag{4}$$

where the primed summation means any terms where the denominator is zero are ignored.

The proof of Theorem 2 is given below in Section 3 and essentially relies on the PNT. From this theorem and the fact that $\pi_k(n) \ll n/\log^k n$ we have the following aesthetically pleasing corollary.

Corollary 3. *For any integer $k \geq 1$,*

$$\sum'_{n \leq x} \frac{1}{\pi_k(n)} \ll_k \frac{x}{\pi_{k+1}(x)}.$$

Note that this corollary can be interpreted as saying the average value of $1/\pi_k(n)$ on $[1, x]$ is bounded by a constant multiple of $1/\pi_{k+1}(x)$. Lastly, we give a similar theorem involving the von Mangoldt function $\Lambda_k(n) := \Lambda_{\mathbb{P}_k}(n)$.

Theorem 4. *For any integer $k \geq 1$ and a fixed integer $N > k$ the function $\Lambda_k(n)$ satisfies*

$$\sum'_{\substack{n \leq x \\ n \in \mathbb{N}_k}} \frac{1}{\Lambda_k(n)} = \sum_{j=k+1}^N \frac{c_j x}{\log^j x} + O_k\left(\frac{x \log \log x}{\log^{k+2} x}\right), \tag{5}$$

where $c_{k+1} = 1$ and the remaining c_j are computable constants.

The proof relies on the following bound for the generalized Chebyshev theta function $\theta_k(x) := \theta_{\mathbb{P}_k}(x)$.

Lemma 5. *For any integer $k \geq 1$ and a fixed positive integer N the function $\theta_k(x)$ satisfies*

$$\theta_k(x) = \frac{x}{\log^{k-1} x} - \sum_{j=k}^N \frac{(j-1)!}{(k-1)! \log^j x} x + O_k\left(\frac{x \log \log x}{\log^k x}\right), \tag{6}$$

where the summation is understood to be zero if $N < k$.

Both proofs of Theorem 4 and Lemma 5 are given below in Section 4.

2. Proof of Theorem 1

The basis of the theorem is the following bound for the n -th prime.

Lemma 6. *The function $\pi_{-1}(n)$ satisfies*

$$\frac{\pi_{-1}(n)}{n} = \log n + \log \log n - 1 + O\left(\frac{\log \log n}{\log n}\right).$$

Proof. This follows from a bound of Pervouchine which can be found in [Cesàro 1894]. □

Next, we give a lemma that will be used to bound compositions of $\pi_{-1}(n)$.

Lemma 7. Let $f(n, j)$ be defined as

$$f(n, j) := \log n + j \log \log n - 1.$$

Then for any fixed integers $j, l \geq 1$ the following three equations hold:

$$\log(nf(n, j)) = \log n + \log \log n + O_j\left(\frac{\log \log n}{\log n}\right), \tag{7}$$

$$\log \log(nf(n, j)) = \log \log n + O_j\left(\frac{\log \log n}{\log n}\right), \tag{8}$$

$$f(nf(n, j), l) = f(n, l + 1) + O_{j,l}\left(\frac{\log \log n}{\log n}\right). \tag{9}$$

Proof. The proof is a straightforward computation, but we include it here for completeness. First, using the definition of $f(n, j)$ and properties of logarithms, we have

$$\begin{aligned} \log(nf(n, j)) &= \log n + \log(\log n + j \log \log n - 1) \\ &= \log n + \log\left(\log n \left(1 + j \frac{\log \log n}{\log n} - \frac{1}{\log n}\right)\right) \\ &= \log n + \log \log n + \log\left(1 + j \frac{\log \log n}{\log n} - \frac{1}{\log n}\right). \end{aligned}$$

Now (7) follows since

$$\log\left(1 + j \frac{\log \log n}{\log n} - \frac{1}{\log n}\right) \ll_j \frac{\log \log n}{\log n}$$

by expanding the logarithm. Similarly, from (7) we obtain

$$\begin{aligned} \log \log(nf(n, j)) &= \log\left(\log n + \log \log n + O_j\left(\frac{\log \log n}{\log n}\right)\right) \\ &= \log\left(\log n \left(1 + \frac{\log \log n}{\log n} + O_j\left(\frac{\log \log n}{\log^2 n}\right)\right)\right) \\ &= \log \log n + \log\left(1 + \frac{\log \log n}{\log n} + O_j\left(\frac{\log \log n}{\log^2 n}\right)\right), \end{aligned}$$

and (8) follows from expanding the last term. Finally, (9) follows immediately from the definition of f and (7) and (8):

$$\begin{aligned} f(nf(n, j), l) &= \log(nf(n, j)) + l \log \log(nf(n, j)) - 1 \\ &= \log n + \log \log n + O_j\left(\frac{\log \log n}{\log n}\right) + l \left(\log \log n + O_j\left(\frac{\log \log n}{\log n}\right)\right) - 1 \\ &= \log n + (l + 1) \log \log n - 1 + O_{j,l}\left(\frac{\log \log n}{\log n}\right). \quad \square \end{aligned}$$

Remark. The three equations in Lemma 7 remain valid if we replace $f(n, j)$ by $f(n, j) + O(\log \log n / \log n)$ as the computations remain unchanged.

Proof of Theorem 1. We proceed via induction on k . Since the $k = 1$ case reduces to Lemma 6, suppose (3) holds for some $k = m > 1$; that is,

$$\pi_{-m}(n) = n \prod_{j=1}^m f(n, j) + O_m(n \log^{m-2} n \log \log n).$$

Recall that $\pi_{-(m+1)} = \pi_{-m} \circ \pi_{-1}$, so we have

$$\begin{aligned} \pi_{-(m+1)}(n) &= \pi_{-1}(n) \prod_{j=1}^m f(\pi_{-1}(n), j) + O_m(\pi_{-1}(n) \log^{m-2}(\pi_{-1}(n)) \log \log(\pi_{-1}(n))). \end{aligned}$$

Note that by Lemmas 6 and 7

$$f(\pi_{-1}(n), l) = f\left(nf(n, 1) + O\left(\frac{n \log \log n}{\log n}\right), l\right) = f(n, l+1) + O_l\left(\frac{\log \log n}{\log n}\right)$$

for any $l \geq 1$. Applying this and Lemma 6 to the expression for $\pi_{-(m+1)}(n)$ above yields

$$\begin{aligned} \pi_{-(m+1)}(n) &= n \prod_{j=1}^{m+1} \left(f(n, j) + O_j\left(\frac{\log \log n}{\log n}\right) \right) \\ &\quad + O_m(nf(n, 1) \log^{m-2}(nf(n, 1)) \log \log(nf(n, 1))). \end{aligned}$$

The error terms arising from the product are at most

$$n \log^m n O_m\left(\frac{\log \log n}{\log n}\right) = O_m(n \log^{m-1} n \log \log n),$$

and by Lemma 7 we have

$$nf(n, 1) \log^{m-2}(nf(n, 1)) \log \log(nf(n, 1)) \ll n \log^{m-1} n \log \log n.$$

Thus,

$$\pi_{-(m+1)}(n) = n \prod_{j=1}^{m+1} f(n, j) + O_m(n \log^{m-1} n \log \log n),$$

which completes the proof. □

3. Proof of Theorem 2

The proof depends on the following two lemmas.

Lemma 8. For any integer $k \geq 1$ the function $\pi_k(n)$ satisfies

$$\pi_k(n) = \frac{n}{\log^k n} + O_k\left(\frac{n \log \log n}{\log^{k+1} n}\right). \tag{10}$$

Proof. This follows from the PNT; see Theorem 7.1 in [Bayless et al. 2013]. \square

Lemma 9. For any integer $k \geq 1$ we have

$$\sum_{n \leq x} \frac{\log^k n}{n} = \frac{\log^{k+1} x}{k+1} + O_k(1). \tag{11}$$

Proof. Recall Abel’s summation formula²

$$\sum_{y < n \leq x} a_n \phi(n) = A(x)\phi(x) - A(y)\phi(y) - \int_y^x A(t)\phi'(t) dt, \tag{12}$$

where $\phi \in C^1$ and $A(x) = \sum_{n \leq x} a_n$ for some sequence (a_n) . Taking $a_n \equiv 1$ we have $A(x) = [x] = x - \{x\}$, where $\{x\}$ denotes the fractional part of x , and so

$$\sum_{y < n \leq x} \phi(n) = (x - \{x\})\phi(x) - (y - \{y\})\phi(y) - \int_y^x (t - \{t\})\phi'(t) dt.$$

Observing that $\int_y^x t\phi'(t) dt = x\phi(x) - y\phi(y) - \int_y^x \phi(t) dt$, we arrive at the following form of Abel’s summation formula:

$$\sum_{y < n \leq x} \phi(n) = \int_y^x \phi(t) dt - \{x\}\phi(x) + \{y\}\phi(y) + \int_y^x \{t\}\phi'(t) dt.$$

Setting $y = 1$ and $\phi(x) = \log^k x/x$ yields

$$\sum_{n \leq x} \frac{\log^k n}{n} = \frac{\log^{k+1} x}{k+1} - \{x\} \frac{\log^k x}{x} + \int_1^x \{t\}\phi'(t) dt,$$

or, in other words,

$$\sum_{n \leq x} \frac{\log^k n}{n} - \frac{\log^{k+1} x}{k+1} \ll \phi(x) + \int_1^x |\phi'(t)| dt. \tag{13}$$

Now $\phi'(x) > 0$ on the interval $(1, e^k)$, and $\phi'(x) < 0$ on (e^k, ∞) . Thus, by the fundamental theorem of calculus, the right-hand side of (13) becomes

$$\phi(x) + \phi(e^k) - \phi(1) - \phi(x) + \phi(e^k) = 2\phi(e^k) = 2\left(\frac{k}{e}\right)^k,$$

which proves the lemma. \square

²Abel’s summation formula is integration by parts of a Riemann–Stieltjes integral; see Theorem 4.2 in [Apostol 1976] and Appendix A in [Montgomery and Vaughan 2007]. Several of the computations here, especially those in Section 4 below, become more straightforward with this machinery.

Remark. Lemma 9 also follows from the fact that

$$\lim_{x \rightarrow \infty} \left(\sum_{n \leq x} \frac{\log^k n}{n} - \frac{\log^{k+1} x}{k+1} \right) = \gamma_k,$$

where γ_k are constants, called Stieltjes constants, that occur in the Laurent series expansion of the Riemann zeta function about $s = 1$; in particular,

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_k (s-1)^k;$$

see Example 1.3.12 on p. 29 of [Montgomery and Vaughan 2007]. Consequently, the proof of Lemma 9 also gives a simple upper bound for $|\gamma_k|$.

Proof of Theorem 2. The $k = 1$ case is Theorem 9.1 in [De Koninck and Ivić 1980], so assume $k \geq 2$. From Lemmas 8 and 9 we have

$$\sum'_{n \leq x} \left(\frac{1}{\pi_k(n)} - \frac{1}{n/\log^k n} \right) = O_k \left(\sum'_{n \leq x} \frac{\log \log n \log^{k-1} n}{n} \right).$$

The theorem now follows from the fact that

$$\begin{aligned} \sum'_{n \leq x} \frac{\log \log n \log^{k-1} n}{n} &\ll \int_1^x \frac{\log \log t \log^{k-1} t}{t} dt \\ &= \log \log t \frac{\log^k t}{k} \Big|_1^x - \frac{1}{k} \int_1^x \frac{\log^{k-1} t}{t} dt \\ &= \frac{1}{k} \log \log x \log^k x - \frac{\log^k x}{k^2}. \quad \square \end{aligned}$$

4. Proofs of Theorem 4 and Lemma 5

We first prove the bound for the Chebyshev theta function over \mathbb{P}_k .

Proof of Lemma 5. Using Riemann–Stieltjes integration, or Abel’s summation formula in (12) with $\phi(x) = \log x$ and $A(x) = \pi_k(x)$, we have

$$\theta_k(x) = \int_2^x \log x \, d\pi_k(x) = \pi_k(x) \log x - \int_2^x \frac{\pi_k(t)}{t} dt$$

(note this reduces to Theorem 4.3 in [Apostol 1976] when $k = 1$). By Lemma 8 this becomes

$$\theta_k(x) = \frac{x}{\log^{k-1} x} + O_k \left(\frac{x \log \log x}{\log^k x} \right) - \int_2^x \frac{dt}{\log^k t} - O_k \left(\int_2^x \frac{\log \log t}{\log^{k+1} t} dt \right). \quad (14)$$

Next, integration by parts yields the formula

$$\int \frac{dt}{\log^k t} = \frac{t}{\log^k t} + k \int \frac{dt}{\log^{k+1} t},$$

which through repeated use gives

$$\int_2^x \frac{dt}{\log^k t} = \sum_{j=k}^N \frac{(j-1)!}{(k-1)!} \frac{x}{\log^j x} + O_k\left(\frac{x}{\log^{N+1} x}\right). \tag{15}$$

Similarly, using integration by parts on the integral in the error term, we have

$$\int \frac{\log \log t}{\log^{k+1} t} dt = \frac{t \log \log t}{\log^{k+1} t} - \int \frac{dt}{\log^{k+2} t} + (k+1) \int \frac{\log \log t}{\log^{k+2} t} dt,$$

which implies

$$\int_2^x \frac{\log \log t}{\log^{k+1} t} dt \ll \frac{x \log \log x}{\log^{k+1} x} \ll \frac{x \log \log x}{\log^k x}. \tag{16}$$

The lemma now follows from using (16) and (15) in (14). □

Proof of Theorem 4. The $k = 1$ case is Theorem 9.2 in [De Koninck and Ivić 1980], so assume $k \geq 2$. The idea of the proof is to use the definition of Λ_k to write

$$\sum'_{\substack{n \leq x \\ n \in \mathbb{N}_k}} \frac{1}{\Lambda_k(n)} = \sum_{\substack{p^\alpha \leq x \\ \alpha \geq 1 \\ p \in \mathbb{P}_k}} \frac{1}{\log p} = \sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} + \sum_{\substack{p^2 \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} + \dots$$

and then bound the summations on the right-hand side.³ To this end, we again use Riemann–Stieltjes integration, or Abel’s summation formula in (12) with $\phi(x) = 1/\log^2 x$ and $A(x) = \theta_k(x)$, to obtain

$$\sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} = \int_2^x \frac{1}{\log^2 t} d\theta_k(t) = \frac{\theta_k(x)}{\log^2 x} + 2 \int_2^x \frac{\theta_k(t)}{t \log^3 t} dt,$$

which by Lemma 5 becomes

$$\begin{aligned} \sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} &= \frac{x}{\log^{k+1} x} - \sum_{j=k}^{N-2} \frac{(j-1)!}{(k-1)!} \frac{x}{\log^{j+2} x} + O_k\left(\frac{x \log \log x}{\log^{k+2} x}\right) \\ &+ 2 \int_2^x \frac{dt}{\log^{k+2} t} - 2 \sum_{j=k}^{N-3} \frac{(j-1)!}{(k-1)!} \int_2^x \frac{dt}{\log^{j+3} t} + O_k\left(\int_2^x \frac{\log \log t}{\log^{k+3} t} dt\right). \end{aligned} \tag{17}$$

³Note that there are only finitely many summations for a given x because $p^\alpha \leq x$ implies $\alpha \leq \log x / \log p \leq \log x / \log 2$.

Note that the upper bounds on the summations have been chosen so that they terminate on the order of $x / \log^N x$. We can combine error terms by taking $k \rightarrow k+2$ in (16); that is,

$$\int_2^x \frac{\log \log t}{\log^{k+3} t} dt \ll \frac{x \log \log x}{\log^{k+3} x} \ll \frac{x \log \log x}{\log^{k+2} x}. \tag{18}$$

Now by (15) we have

$$\begin{aligned} \int_2^x \frac{dt}{\log^{k+2} t} &= \sum_{j=k+2}^N \frac{(j-1)!}{(k+1)!} \frac{x}{\log^j x} + O_k\left(\frac{x}{\log^{N+1} x}\right) \\ &= \sum_{j=k}^{N-2} \frac{(j+1)!}{(k+1)!} \frac{x}{\log^{j+2} x} + O_k\left(\frac{x}{\log^{N+1} x}\right), \end{aligned} \tag{19}$$

as well as

$$\int_2^x \frac{dt}{\log^{j+3} t} = \frac{x}{\log^{j+3} x} + O_j\left(\frac{x}{\log^{j+4} x}\right). \tag{20}$$

Putting (18)–(20) into (17) yields

$$\begin{aligned} \sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} &= \frac{x}{\log^{k+1} x} + \sum_{j=k}^{N-2} \left(2 \frac{(j+1)!}{(k+1)!} - \frac{(j-1)!}{(k-1)!}\right) \frac{x}{\log^{j+2} x} \\ &\quad - 2 \sum_{j=k}^{N-3} \frac{(j-1)!}{(k-1)!} \frac{x}{\log^{j+3} x} + O_k\left(\frac{x \log \log x}{\log^{k+2} x}\right), \end{aligned}$$

which can be rewritten as

$$\sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} = \sum_{j=k+1}^N \frac{c_j x}{\log^j x} + O_k\left(\frac{x \log \log x}{\log^{k+2} x}\right).$$

The theorem now follows from the fact that

$$\sum'_{\substack{n \leq x \\ n \in \mathbb{N}_k}} \frac{1}{\Lambda_k(n)} = \sum_{\substack{p^\alpha \leq x \\ \alpha \geq 1 \\ p \in \mathbb{P}_k}} \frac{1}{\log p} = \sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} + o\left(\sum_{\substack{p^2 \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p}\right),$$

and by observing that

$$\sum_{\substack{p \leq \sqrt{x} \\ p \in \mathbb{P}_k}} \frac{1}{\log p} \ll \frac{\sqrt{x}}{\log^k \sqrt{x}} \ll \frac{x \log \log x}{\log^{k+2} x}$$

by Lemma 8. □

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Received: 2018-08-29

Revised: 2019-07-25

Accepted: 2019-12-28

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Cover: Alex Scorpan

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

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2020 vol. 13 no. 2

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