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The sieve of Eratosthenes (SoE) is a well-known method of extracting the set of prime numbers  $\mathbb{P}$  from the set positive integers  $\mathbb{N}$ . Applying the SoE again to the index of the prime numbers will result in the set of prime-indexed primes  $\mathbb{P}_2 = \{3, 5, 11, 17, 31, \dots\}$ . More generally, the application of the SoE  $k$ -times will yield the set  $\mathbb{P}_k$  of  $k$ -th order primes. In this paper, we give an upper bound for the  $n$ -th  $k$ -order prime as well as some results relating to number-theoretic functions over  $\mathbb{P}_k$ .

## 1. Introduction

This paper lies within the intersection of two topics in analytic number theory: abstract analytic number theory and the prime-indexed primes. In this section we give brief summaries of these topics followed by our results, whose proofs are presented in the subsequent sections.

**1.1. Abstract analytic number theory.** We begin with a set of *generalized primes*  $\mathcal{P} := \{p_i \in \mathbb{R} \mid 1 < p_1 \leq p_2 \leq \dots < \infty\}$  out of which the set of *generalized integers*  $\mathcal{N} := \{n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \mid p_i \in \mathcal{P}, \alpha_i \in \mathbb{N}\}$  is constructed; that is,  $\mathcal{N}$  consists of all possible finite words over  $\mathcal{P}$ . One of the main goals of abstract analytic number theory is to describe the asymptotic behavior of, and the relationship between, the two counting functions

$$\pi_{\mathcal{P}}(x) := \sum_{\substack{p < x \\ p \in \mathcal{P}}} 1 \quad \text{and} \quad N_{\mathcal{P}}(x) := \sum_{\substack{n < x \\ n \in \mathcal{N}}} 1. \quad (1)$$

For example, what growth conditions must we impose on  $N_{\mathcal{P}}(x)$  so that  $\pi_{\mathcal{P}}(x)$  satisfies the classical prime number theorem (PNT)  $\pi_{\mathcal{P}}(x) \sim x / \log x$ ? One might naturally suppose that  $N_{\mathcal{P}}(x) \sim x$  may be enough to guarantee the PNT, but Beurling [1937] showed this is not the case. In particular, he proved that if  $N_{\mathcal{P}}(x) = Ax + O(x / \log^\gamma x)$ ,  $A > 0$ , and  $\gamma > \frac{3}{2}$ , then the PNT follows; however, he also showed

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the PNT can fail if  $\gamma = \frac{3}{2}$ . For more on the PNT and Beurling's generalization of it, we refer the reader to Sections 1.1, 6.2, and 8.4 of [Montgomery and Vaughan 2007].

Central to the proof of Beurling's generalized PNT, and arguably the most fundamental object in abstract analytic number theory, is the *Beurling zeta function*  $\zeta_{\mathcal{P}}(s)$ , which is defined as

$$\zeta_{\mathcal{P}}(s) := \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n \in \mathcal{N}} \frac{1}{n^s} \quad (2)$$

for  $s \in \mathbb{C}$  wherever it converges.<sup>1</sup> Classically, many important arithmetic functions arise as the coefficients of Dirichlet series involving the Riemann zeta function  $\zeta(s) := \zeta_{\mathbb{P}}(s)$ ; see Chapter 1 of [Titchmarsh 1986]. The von Mangoldt function  $\Lambda(n)$ , for instance, satisfies

$$\frac{\zeta'(s)}{\zeta(s)} = - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \text{where } \Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Replacing the Riemann zeta function with the Beurling zeta function, we arrive at the generalized von Mangoldt function  $\Lambda_{\mathcal{P}}(n)$  defined over  $\mathcal{N}$ ; that is,

$$\frac{\zeta'_{\mathcal{P}}(s)}{\zeta_{\mathcal{P}}(s)} = - \sum_{n \in \mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^s}, \quad \text{where } \Lambda_{\mathcal{P}}(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for } p \in \mathcal{P}, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper we will restrict our attention to the generalized prime counting function  $\pi_{\mathcal{P}}(n)$ , the  $n$ -th prime function  $\pi_{\mathcal{P}}^{-1}(n)$ , the von Mangoldt function  $\Lambda_{\mathcal{P}}(n)$ , and the Chebyshev theta function

$$\theta_{\mathcal{P}}(n) := \sum_{\substack{p < x \\ p \in \mathcal{P}}} \log p.$$

For further information on the Beurling zeta function and the various arithmetic functions associated to it, see the excellent summary given in Chapter 1 of [Diamond and Zhang 2016], as well as the more detailed Chapter 2 of [Knopfmacher 1990].

**1.2. The  $k$ -th order primes.** The sieve of Eratosthenes (SoE) is a well-known method of extracting the set of prime numbers  $\mathbb{P}$  from the set of positive integers  $\mathbb{N}$ . Applying the SoE again to the index of the prime numbers will result in the set of prime-indexed primes  $\mathbb{P}_2 = \{3, 5, 11, 17, 31, 41, 59, 67, 83, \dots\}$ . More generally, the application of the SoE  $k$ -times will yield the set  $\mathbb{P}_k$  of  $k$ -th order primes. Regarding  $\mathbb{P}_k$  as a set of generalized primes, one also has the corresponding set  $\mathbb{N}_k$

<sup>1</sup>It is known (see Proposition 2.1 in Chapter 4 of [Knopfmacher 1990]) that the Euler product and the Dirichlet series converge in the same half-plane and are equal.

of generalized integers (as constructed above in [Section 1.1](#)). It follows by definition that the corresponding prime counting functions are given by

$$\pi_k(x) := \pi_{\mathbb{P}_k}(x) = \underbrace{(\pi \circ \cdots \circ \pi)}_{k\text{-times}}(x),$$

and the PNT implies  $\pi_k(x) \sim x / \log^k x$  (see [Lemma 8](#) below). Furthermore, if we set  $\pi^{-1}(n) := p_n$ , where  $p_n$  is the  $n$ -th prime, then the  $n$ -th  $k$ -order prime is given by

$$\pi_{-k}(n) := \pi_{\mathbb{P}_k}^{-1}(n) = \underbrace{(\pi^{-1} \circ \cdots \circ \pi^{-1})}_{k\text{-times}}(n).$$

The  $k$ -th order primes  $\mathbb{P}_k$  and corresponding functions  $\pi_{\pm k}(n)$  have been studied recently in [[Broughan and Barnett 2009](#); [Bayless et al. 2013](#)], with emphasis on the  $k = 2$ , prime-indexed prime case. In addition to generalizing fundamental questions like the Goldbach and twin prime conjectures to  $\mathbb{P}_2$ , bounds have been derived for  $\pi_{\pm 2}(n)$ , as well as  $\pi_k(n)$  for  $k \geq 2$ . For instance, Broughan and Barnett give the bounds

$$\begin{aligned} \pi_{-2}(n) &< n(\log n + \log \log n)(\log n + 2 \log \log n) - n \log n + O(n \log \log n), \\ \pi_{-2}(n) &> n(\log n + \log \log n)(\log n + 2 \log \log n) - 3n \log n + O(n \log \log n). \end{aligned}$$

The goal of this paper is to continue this line of inquiry against the backdrop of abstract analytic number theory.

**1.3. Results.** Our main result is a bound for  $\pi_{-k}(n)$  for all  $k \geq 1$ . Consequently, the following theorem, together with the PNT, bound  $\pi_k(n)$  for all  $k \in \mathbb{Z}$ .

**Theorem 1.** *For any integer  $k \geq 1$  the function  $\pi_{-k}(n)$  satisfies*

$$\pi_{-k}(n) = n \prod_{j=1}^k f(n, j) + O_k(n \log^{k-2} n \log \log n), \quad (3)$$

where  $f(n, j) := \log n + j \log \log n - 1$ .

The proof of [Theorem 1](#) is given below in [Section 2](#). Ideally, one would like to determine the precise relation between  $k$  and the error constants in the theorem.

The next two theorems are generalizations of [Theorems 9.1 and 9.2](#) in [[De Koninck and Ivić 1980](#)] involving sums of arithmetic functions. The first involves the  $k$ -th order prime counting function  $\pi_k(n)$ .

**Theorem 2.** *For any integer  $k \geq 1$  the function  $\pi_k(x)$  satisfies*

$$\sum'_{n \leq x} \frac{1}{\pi_k(n)} = \frac{\log^{k+1} x}{k+1} + O_k(\log^k x \log \log x), \quad (4)$$

where the primed summation means any terms where the denominator is zero are ignored.

The proof of [Theorem 2](#) is given below in [Section 3](#) and essentially relies on the PNT. From this theorem and the fact that  $\pi_k(n) \ll n / \log^k n$  we have the following aesthetically pleasing corollary.

**Corollary 3.** *For any integer  $k \geq 1$ ,*

$$\sum'_{n \leq x} \frac{1}{\pi_k(n)} \ll_k \frac{x}{\pi_{k+1}(x)}.$$

Note that this corollary can be interpreted as saying the average value of  $1/\pi_k(n)$  on  $[1, x]$  is bounded by a constant multiple of  $1/\pi_{k+1}(x)$ . Lastly, we give a similar theorem involving the von Mangoldt function  $\Lambda_k(n) := \Lambda_{\mathbb{P}_k}(n)$ .

**Theorem 4.** *For any integer  $k \geq 1$  and a fixed integer  $N > k$  the function  $\Lambda_k(n)$  satisfies*

$$\sum'_{\substack{n \leq x \\ n \in \mathbb{N}_k}} \frac{1}{\Lambda_k(n)} = \sum_{j=k+1}^N \frac{c_j x}{\log^j x} + O_k \left( \frac{x \log \log x}{\log^{k+2} x} \right), \quad (5)$$

where  $c_{k+1} = 1$  and the remaining  $c_j$  are computable constants.

The proof relies on the following bound for the generalized Chebyshev theta function  $\theta_k(x) := \theta_{\mathbb{P}_k}(x)$ .

**Lemma 5.** *For any integer  $k \geq 1$  and a fixed positive integer  $N$  the function  $\theta_k(x)$  satisfies*

$$\theta_k(x) = \frac{x}{\log^{k-1} x} - \sum_{j=k}^N \frac{(j-1)!}{(k-1)!} \frac{x}{\log^j x} + O_k \left( \frac{x \log \log x}{\log^k x} \right), \quad (6)$$

where the summation is understood to be zero if  $N < k$ .

Both proofs of [Theorem 4](#) and [Lemma 5](#) are given below in [Section 4](#).

## 2. Proof of [Theorem 1](#)

The basis of the theorem is the following bound for the  $n$ -th prime.

**Lemma 6.** *The function  $\pi_{-1}(n)$  satisfies*

$$\frac{\pi_{-1}(n)}{n} = \log n + \log \log n - 1 + O \left( \frac{\log \log n}{\log n} \right).$$

*Proof.* This follows from a bound of Pervouchine which can be found in [\[Cesàro 1894\]](#). □

Next, we give a lemma that will be used to bound compositions of  $\pi_{-1}(n)$ .

**Lemma 7.** Let  $f(n, j)$  be defined as

$$f(n, j) := \log n + j \log \log n - 1.$$

Then for any fixed integers  $j, l \geq 1$  the following three equations hold:

$$\log(nf(n, j)) = \log n + \log \log n + O_j\left(\frac{\log \log n}{\log n}\right), \quad (7)$$

$$\log \log(nf(n, j)) = \log \log n + O_j\left(\frac{\log \log n}{\log n}\right), \quad (8)$$

$$f(nf(n, j), l) = f(n, l+1) + O_{j,l}\left(\frac{\log \log n}{\log n}\right). \quad (9)$$

*Proof.* The proof is a straightforward computation, but we include it here for completeness. First, using the definition of  $f(n, j)$  and properties of logarithms, we have

$$\begin{aligned} \log(nf(n, j)) &= \log n + \log(\log n + j \log \log n - 1) \\ &= \log n + \log\left(\log n \left(1 + j \frac{\log \log n}{\log n} - \frac{1}{\log n}\right)\right) \\ &= \log n + \log \log n + \log\left(1 + j \frac{\log \log n}{\log n} - \frac{1}{\log n}\right). \end{aligned}$$

Now (7) follows since

$$\log\left(1 + j \frac{\log \log n}{\log n} - \frac{1}{\log n}\right) \ll_j \frac{\log \log n}{\log n}$$

by expanding the logarithm. Similarly, from (7) we obtain

$$\begin{aligned} \log \log(nf(n, j)) &= \log\left(\log n + \log \log n + O_j\left(\frac{\log \log n}{\log n}\right)\right) \\ &= \log\left(\log n \left(1 + \frac{\log \log n}{\log n} + O_j\left(\frac{\log \log n}{\log^2 n}\right)\right)\right) \\ &= \log \log n + \log\left(1 + \frac{\log \log n}{\log n} + O_j\left(\frac{\log \log n}{\log^2 n}\right)\right), \end{aligned}$$

and (8) follows from expanding the last term. Finally, (9) follows immediately from the definition of  $f$  and (7) and (8):

$$\begin{aligned} f(nf(n, j), l) &= \log(nf(n, j)) + l \log \log(nf(n, j)) - 1 \\ &= \log n + \log \log n + O_j\left(\frac{\log \log n}{\log n}\right) + l \left(\log \log n + O_j\left(\frac{\log \log n}{\log n}\right)\right) - 1 \\ &= \log n + (l+1) \log \log n - 1 + O_{j,l}\left(\frac{\log \log n}{\log n}\right). \quad \square \end{aligned}$$

**Remark.** The three equations in [Lemma 7](#) remain valid if we replace  $f(n, j)$  by  $f(n, j) + O(\log \log n / \log n)$  as the computations remain unchanged.

*Proof of [Theorem 1](#).* We proceed via induction on  $k$ . Since the  $k = 1$  case reduces to [Lemma 6](#), suppose [\(3\)](#) holds for some  $k = m > 1$ ; that is,

$$\pi_{-m}(n) = n \prod_{j=1}^m f(n, j) + O_m(n \log^{m-2} n \log \log n).$$

Recall that  $\pi_{-(m+1)} = \pi_{-m} \circ \pi_{-1}$ , so we have

$$\begin{aligned} \pi_{-(m+1)}(n) &= \pi_{-1}(n) \prod_{j=1}^m f(\pi_{-1}(n), j) + O_m(\pi_{-1}(n) \log^{m-2}(\pi_{-1}(n)) \log \log(\pi_{-1}(n))). \end{aligned}$$

Note that by [Lemmas 6](#) and [7](#)

$$f(\pi_{-1}(n), l) = f\left(nf(n, 1) + O\left(\frac{n \log \log n}{\log n}\right), l\right) = f(n, l+1) + O_l\left(\frac{\log \log n}{\log n}\right)$$

for any  $l \geq 1$ . Applying this and [Lemma 6](#) to the expression for  $\pi_{-(m+1)}(n)$  above yields

$$\begin{aligned} \pi_{-(m+1)}(n) &= n \prod_{j=1}^{m+1} \left( f(n, j) + O_j\left(\frac{\log \log n}{\log n}\right) \right) \\ &\quad + O_m(nf(n, 1) \log^{m-2}(nf(n, 1)) \log \log(nf(n, 1))). \end{aligned}$$

The error terms arising from the product are at most

$$n \log^m n O_m\left(\frac{\log \log n}{\log n}\right) = O_m(n \log^{m-1} n \log \log n),$$

and by [Lemma 7](#) we have

$$nf(n, 1) \log^{m-2}(nf(n, 1)) \log \log(nf(n, 1)) \ll n \log^{m-1} n \log \log n.$$

Thus,

$$\pi_{-(m+1)}(n) = n \prod_{j=1}^{m+1} f(n, j) + O_m(n \log^{m-1} n \log \log n),$$

which completes the proof. □

### 3. Proof of [Theorem 2](#)

The proof depends on the following two lemmas.

**Lemma 8.** For any integer  $k \geq 1$  the function  $\pi_k(n)$  satisfies

$$\pi_k(n) = \frac{n}{\log^k n} + O_k\left(\frac{n \log \log n}{\log^{k+1} n}\right). \quad (10)$$

*Proof.* This follows from the PNT; see Theorem 7.1 in [Bayless et al. 2013].  $\square$

**Lemma 9.** For any integer  $k \geq 1$  we have

$$\sum_{n \leq x} \frac{\log^k n}{n} = \frac{\log^{k+1} x}{k+1} + O_k(1). \quad (11)$$

*Proof.* Recall Abel's summation formula<sup>2</sup>

$$\sum_{y < n \leq x} a_n \phi(n) = A(x)\phi(x) - A(y)\phi(y) - \int_y^x A(t)\phi'(t) dt, \quad (12)$$

where  $\phi \in \mathcal{C}^1$  and  $A(x) = \sum_{n \leq x} a_n$  for some sequence  $(a_n)$ . Taking  $a_n \equiv 1$  we have  $A(x) = \lfloor x \rfloor = x - \{x\}$ , where  $\{x\}$  denotes the fractional part of  $x$ , and so

$$\sum_{y < n \leq x} \phi(n) = (x - \{x\})\phi(x) - (y - \{y\})\phi(y) - \int_y^x (t - \{t\})\phi'(t) dt.$$

Observing that  $\int_y^x t\phi'(t) dt = x\phi(x) - y\phi(y) - \int_y^x \phi(t) dt$ , we arrive at the following form of Abel's summation formula:

$$\sum_{y < n \leq x} \phi(n) = \int_y^x \phi(t) dt - \{x\}\phi(x) + \{y\}\phi(y) + \int_y^x \{t\}\phi'(t) dt.$$

Setting  $y = 1$  and  $\phi(x) = \log^k x/x$  yields

$$\sum_{n \leq x} \frac{\log^k n}{n} = \frac{\log^{k+1} x}{k+1} - \{x\} \frac{\log^k x}{x} + \int_1^x \{t\}\phi'(t) dt,$$

or, in other words,

$$\sum_{n \leq x} \frac{\log^k n}{n} - \frac{\log^{k+1} x}{k+1} \ll \phi(x) + \int_1^x |\phi'(t)| dt. \quad (13)$$

Now  $\phi'(x) > 0$  on the interval  $(1, e^k)$ , and  $\phi'(x) < 0$  on  $(e^k, \infty)$ . Thus, by the fundamental theorem of calculus, the right-hand side of (13) becomes

$$\phi(x) + \phi(e^k) - \phi(1) - \phi(x) + \phi(e^k) = 2\phi(e^k) = 2\left(\frac{k}{e}\right)^k,$$

which proves the lemma.  $\square$

<sup>2</sup>Abel's summation formula is integration by parts of a Riemann–Stieltjes integral; see Theorem 4.2 in [Apostol 1976] and Appendix A in [Montgomery and Vaughan 2007]. Several of the computations here, especially those in Section 4 below, become more straightforward with this machinery.

**Remark.** Lemma 9 also follows from the fact that

$$\lim_{x \rightarrow \infty} \left( \sum_{n \leq x} \frac{\log^k n}{n} - \frac{\log^{k+1} x}{k+1} \right) = \gamma_k,$$

where  $\gamma_k$  are constants, called Stieltjes constants, that occur in the Laurent series expansion of the Riemann zeta function about  $s = 1$ ; in particular,

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_k (s-1)^k;$$

see Example 1.3.12 on p. 29 of [Montgomery and Vaughan 2007]. Consequently, the proof of Lemma 9 also gives a simple upper bound for  $|\gamma_k|$ .

*Proof of Theorem 2.* The  $k = 1$  case is Theorem 9.1 in [De Koninck and Ivić 1980], so assume  $k \geq 2$ . From Lemmas 8 and 9 we have

$$\sum'_{n \leq x} \left( \frac{1}{\pi_k(n)} - \frac{1}{n/\log^k n} \right) = O_k \left( \sum'_{n \leq x} \frac{\log \log n \log^{k-1} n}{n} \right).$$

The theorem now follows from the fact that

$$\begin{aligned} \sum'_{n \leq x} \frac{\log \log n \log^{k-1} n}{n} &\ll \int_1^x \frac{\log \log t \log^{k-1} t}{t} dt \\ &= \log \log t \frac{\log^k t}{k} \Big|_1^x - \frac{1}{k} \int_1^x \frac{\log^{k-1} t}{t} dt \\ &= \frac{1}{k} \log \log x \log^k x - \frac{\log^k x}{k^2}. \end{aligned} \quad \square$$

#### 4. Proofs of Theorem 4 and Lemma 5

We first prove the bound for the Chebyshev theta function over  $\mathbb{P}_k$ .

*Proof of Lemma 5.* Using Riemann–Stieltjes integration, or Abel’s summation formula in (12) with  $\phi(x) = \log x$  and  $A(x) = \pi_k(x)$ , we have

$$\theta_k(x) = \int_2^x \log x \, d\pi_k(x) = \pi_k(x) \log x - \int_2^x \frac{\pi_k(t)}{t} dt$$

(note this reduces to Theorem 4.3 in [Apostol 1976] when  $k = 1$ ). By Lemma 8 this becomes

$$\theta_k(x) = \frac{x}{\log^{k-1} x} + O_k \left( \frac{x \log \log x}{\log^k x} \right) - \int_2^x \frac{dt}{\log^k t} - O_k \left( \int_2^x \frac{\log \log t}{\log^{k+1} t} dt \right). \quad (14)$$

Next, integration by parts yields the formula

$$\int \frac{dt}{\log^k t} = \frac{t}{\log^k t} + k \int \frac{dt}{\log^{k+1} t},$$

which through repeated use gives

$$\int_2^x \frac{dt}{\log^k t} = \sum_{j=k}^N \frac{(j-1)!}{(k-1)!} \frac{x}{\log^j x} + O_k\left(\frac{x}{\log^{N+1} x}\right). \tag{15}$$

Similarly, using integration by parts on the integral in the error term, we have

$$\int \frac{\log \log t}{\log^{k+1} t} dt = \frac{t \log \log t}{\log^{k+1} t} - \int \frac{dt}{\log^{k+2} t} + (k+1) \int \frac{\log \log t}{\log^{k+2} t} dt,$$

which implies

$$\int_2^x \frac{\log \log t}{\log^{k+1} t} dt \ll \frac{x \log \log x}{\log^{k+1} x} \ll \frac{x \log \log x}{\log^k x}. \tag{16}$$

The lemma now follows from using (16) and (15) in (14). □

*Proof of Theorem 4.* The  $k = 1$  case is Theorem 9.2 in [De Koninck and Ivić 1980], so assume  $k \geq 2$ . The idea of the proof is to use the definition of  $\Lambda_k$  to write

$$\sum'_{\substack{n \leq x \\ n \in \mathbb{N}_k}} \frac{1}{\Lambda_k(n)} = \sum_{\substack{p^\alpha \leq x \\ \alpha \geq 1 \\ p \in \mathbb{P}_k}} \frac{1}{\log p} = \sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} + \sum_{\substack{p^2 \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} + \dots$$

and then bound the summations on the right-hand side.<sup>3</sup> To this end, we again use Riemann–Stieltjes integration, or Abel’s summation formula in (12) with  $\phi(x) = 1/\log^2 x$  and  $A(x) = \theta_k(x)$ , to obtain

$$\sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} = \int_2^x \frac{1}{\log^2 t} d\theta_k(t) = \frac{\theta_k(x)}{\log^2 x} + 2 \int_2^x \frac{\theta_k(t)}{t \log^3 t} dt,$$

which by Lemma 5 becomes

$$\begin{aligned} \sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} &= \frac{x}{\log^{k+1} x} - \sum_{j=k}^{N-2} \frac{(j-1)!}{(k-1)!} \frac{x}{\log^{j+2} x} + O_k\left(\frac{x \log \log x}{\log^{k+2} x}\right) \\ &+ 2 \int_2^x \frac{dt}{\log^{k+2} t} - 2 \sum_{j=k}^{N-3} \frac{(j-1)!}{(k-1)!} \int_2^x \frac{dt}{\log^{j+3} t} + O_k\left(\int_2^x \frac{\log \log t}{\log^{k+3} t} dt\right). \end{aligned} \tag{17}$$

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<sup>3</sup>Note that there are only finitely many summations for a given  $x$  because  $p^\alpha \leq x$  implies  $\alpha \leq \log x / \log p \leq \log x / \log 2$ .

Note that the upper bounds on the summations have been chosen so that they terminate on the order of  $x / \log^N x$ . We can combine error terms by taking  $k \rightarrow k+2$  in (16); that is,

$$\int_2^x \frac{\log \log t}{\log^{k+3} t} dt \ll \frac{x \log \log x}{\log^{k+3} x} \ll \frac{x \log \log x}{\log^{k+2} x}. \tag{18}$$

Now by (15) we have

$$\begin{aligned} \int_2^x \frac{dt}{\log^{k+2} t} &= \sum_{j=k+2}^N \frac{(j-1)!}{(k+1)!} \frac{x}{\log^j x} + O_k\left(\frac{x}{\log^{N+1} x}\right) \\ &= \sum_{j=k}^{N-2} \frac{(j+1)!}{(k+1)!} \frac{x}{\log^{j+2} x} + O_k\left(\frac{x}{\log^{N+1} x}\right), \end{aligned} \tag{19}$$

as well as

$$\int_2^x \frac{dt}{\log^{j+3} t} = \frac{x}{\log^{j+3} x} + O_j\left(\frac{x}{\log^{j+4} x}\right). \tag{20}$$

Putting (18)–(20) into (17) yields

$$\begin{aligned} \sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} &= \frac{x}{\log^{k+1} x} + \sum_{j=k}^{N-2} \left(2 \frac{(j+1)!}{(k+1)!} - \frac{(j-1)!}{(k-1)!}\right) \frac{x}{\log^{j+2} x} \\ &\quad - 2 \sum_{j=k}^{N-3} \frac{(j-1)!}{(k-1)!} \frac{x}{\log^{j+3} x} + O_k\left(\frac{x \log \log x}{\log^{k+2} x}\right), \end{aligned}$$

which can be rewritten as

$$\sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} = \sum_{j=k+1}^N \frac{c_j x}{\log^j x} + O_k\left(\frac{x \log \log x}{\log^{k+2} x}\right).$$

The theorem now follows from the fact that

$$\sum'_{\substack{n \leq x \\ n \in \mathbb{N}_k}} \frac{1}{\Lambda_k(n)} = \sum_{\substack{p^\alpha \leq x \\ \alpha \geq 1 \\ p \in \mathbb{P}_k}} \frac{1}{\log p} = \sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} + O\left(\sum_{\substack{p^2 \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p}\right),$$

and by observing that

$$\sum_{\substack{p \leq \sqrt{x} \\ p \in \mathbb{P}_k}} \frac{1}{\log p} \ll \frac{\sqrt{x}}{\log^k \sqrt{x}} \ll \frac{x \log \log x}{\log^{k+2} x}$$

by Lemma 8.

□

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