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# Arithmetic functions of higher-order primes

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The sieve of Eratosthenes (SoE) is a well-known method of extracting the set of prime numbers $\mathbb{P}$ from the set positive integers $\mathbb{N}$. Applying the SoE again to the index of the prime numbers will result in the set of prime-indexed primes $\mathbb{P}_2 = \{3, 5, 11, 17, 31, \ldots\}$. More generally, the application of the SoE $k$-times will yield the set $\mathbb{P}_k$ of $k$-th order primes. In this paper, we give an upper bound for the $n$-th $k$-order prime as well as some results relating to number-theoretic functions over $\mathbb{P}_k$.

## 1. Introduction

This paper lies within the intersection of two topics in analytic number theory: abstract analytic number theory and the prime-indexed primes. In this section we give brief summaries of these topics followed by our results, whose proofs are presented in the subsequent sections.

**1.1.** ***Abstract analytic number theory.*** We begin with a set of *generalized primes* $\mathcal{P} := \{p_i \in \mathbb{R} \mid 1 < p_1 \leq p_2 \leq \cdots < \infty\}$ out of which the set of *generalized integers* $\mathcal{N} := \{n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \mid p_i \in \mathcal{P},\ \alpha_i \in \mathbb{N}\}$ is constructed; that is, $\mathcal{N}$ consists of all possible finite words over $\mathcal{P}$. One of the main goals of abstract analytic number theory is to describe the asymptotic behavior of, and the relationship between, the two counting functions

$$\pi_{\mathcal{P}}(x) := \sum_{\substack{p<x \\ p\in\mathcal{P}}} 1 \quad \text{and} \quad N_{\mathcal{P}}(x) := \sum_{\substack{n<x \\ n\in\mathcal{N}}} 1. \tag{1}$$

For example, what growth conditions must we impose on $N_{\mathcal{P}}(x)$ so that $\pi_{\mathcal{P}}(x)$ satisfies the classical prime number theorem (PNT) $\pi_{\mathcal{P}}(x) \sim x/\log x$? One might naturally suppose that $N_{\mathcal{P}}(x) \sim x$ may be enough to guarantee the PNT, but Beurling [1937] showed this is not the case. In particular, he proved that if $N_{\mathcal{P}}(x) = Ax + O(x/\log^{\gamma} x)$, $A > 0$, and $\gamma > \frac{3}{2}$, then the PNT follows; however, he also showed

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the PNT can fail if $\gamma = \frac{3}{2}$. For more on the PNT and Beurling's generalization of it, we refer the reader to Sections 1.1, 6.2, and 8.4 of [Montgomery and Vaughan 2007].

Central to the proof of Beurling's generalized PNT, and arguably the most fundamental object in abstract analytic number theory, is the *Beurling zeta function* $\zeta_{\mathcal{P}}(s)$, which is defined as

$$\zeta_{\mathcal{P}}(s) := \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n\in\mathcal{N}} \frac{1}{n^s} \tag{2}$$

for $s \in \mathbb{C}$ wherever it converges.[1] Classically, many important arithmetic functions arise as the coefficients of Dirichlet series involving the Riemann zeta function $\zeta(s) := \zeta_{\mathbb{P}}(s)$; see Chapter 1 of [Titchmarsh 1986]. The von Mangoldt function $\Lambda(n)$, for instance, satisfies

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \text{where } \Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Replacing the Riemann zeta function with the Beurling zeta function, we arrive at the generalized von Mangoldt function $\Lambda_{\mathcal{P}}(n)$ defined over $\mathcal{N}$; that is,

$$\frac{\zeta_{\mathcal{P}}'(s)}{\zeta_{\mathcal{P}}(s)} = -\sum_{n\in\mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^s}, \quad \text{where } \Lambda_{\mathcal{P}}(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for } p \in \mathcal{P},\ k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper we will restrict our attention to the generalized prime counting function $\pi_{\mathcal{P}}(n)$, the $n$-th prime function $\pi_{\mathcal{P}}^{-1}(n)$, the von Mangoldt function $\Lambda_{\mathcal{P}}(n)$, and the Chebyshev theta function

$$\theta_{\mathcal{P}}(n) := \sum_{\substack{p<x \\ p\in\mathcal{P}}} \log p.$$

For further information on the Beurling zeta function and the various arithmetic functions associated to it, see the excellent summary given in Chapter 1 of [Diamond and Zhang 2016], as well as the more detailed Chapter 2 of [Knopfmacher 1990].

**1.2.** ***The k-th order primes.*** The sieve of Eratosthenes (SoE) is a well-known method of extracting the set of prime numbers $\mathbb{P}$ from the set of positive integers $\mathbb{N}$. Applying the SoE again to the index of the prime numbers will result in the set of prime-indexed primes $\mathbb{P}_2 = \{3, 5, 11, 17, 31, 41, 59, 67, 83, \ldots\}$. More generally, the application of the SoE $k$-times will yield the set $\mathbb{P}_k$ of $k$-th order primes. Regarding $\mathbb{P}_k$ as a set of generalized primes, one also has the corresponding set $\mathbb{N}_k$

[1] It is known (see Proposition 2.1 in Chapter 4 of [Knopfmacher 1990]) that the Euler product and the Dirichlet series converge in the same half-plane and are equal.

of generalized integers (as constructed above in Section 1.1). It follows by definition that the corresponding prime counting functions are given by

$$\pi_k(x) := \pi_{\mathbb{P}_k}(x) = \underbrace{(\pi \circ \cdots \circ \pi)}_{k-\text{times}}(x),$$

and the PNT implies $\pi_k(x) \sim x/\log^k x$ (see Lemma 8 below). Furthermore, if we set $\pi^{-1}(n) := p_n$, where $p_n$ is the $n$-th prime, then the $n$-th $k$-order prime is given by

$$\pi_{-k}(n) := \pi_{\mathbb{P}_k}^{-1}(n) = \underbrace{(\pi^{-1} \circ \cdots \circ \pi^{-1})}_{k-\text{times}}(n).$$

The $k$-th order primes $\mathbb{P}_k$ and corresponding functions $\pi_{\pm k}(n)$ have been studied recently in [Broughan and Barnett 2009; Bayless et al. 2013], with emphasis on the $k = 2$, prime-indexed prime case. In addition to generalizing fundamental questions like the Goldbach and twin prime conjectures to $\mathbb{P}_2$, bounds have been derived for $\pi_{\pm 2}(n)$, as well as $\pi_k(n)$ for $k \geq 2$. For instance, Broughan and Barnett give the bounds

$$\pi_{-2}(n) < n(\log n + \log\log n)(\log n + 2\log\log n) - n\log n + O(n\log\log n),$$

$$\pi_{-2}(n) > n(\log n + \log\log n)(\log n + 2\log\log n) - 3n\log n + O(n\log\log n).$$

The goal of this paper is to continue this line of inquiry against the backdrop of abstract analytic number theory.

**1.3. *Results.*** Our main result is a bound for $\pi_{-k}(n)$ for all $k \geq 1$. Consequently, the following theorem, together with the PNT, bound $\pi_k(n)$ for all $k \in \mathbb{Z}$.

**Theorem 1.** *For any integer $k \geq 1$ the function $\pi_{-k}(n)$ satisfies*

$$\pi_{-k}(n) = n\prod_{j=1}^{k} f(n, j) + O_k(n\log^{k-2} n \log\log n), \tag{3}$$

*where $f(n, j) := \log n + j\log\log n - 1$.*

The proof of Theorem 1 is given below in Section 2. Ideally, one would like to determine the precise relation between $k$ and the error constants in the theorem.

The next two theorems are generalizations of Theorems 9.1 and 9.2 in [De Koninck and Ivić 1980] involving sums of arithmetic functions. The first involves the $k$-th order prime counting function $\pi_k(n)$.

**Theorem 2.** *For any integer $k \geq 1$ the function $\pi_k(x)$ satisfies*

$$\sideset{}{'}\sum_{n\leq x} \frac{1}{\pi_k(n)} = \frac{\log^{k+1} x}{k+1} + O_k(\log^k x \log\log x), \tag{4}$$

*where the primed summation means any terms where the denominator is zero are ignored.*

The proof of Theorem 2 is given below in Section 3 and essentially relies on the PNT. From this theorem and the fact that $\pi_k(n) \ll n/\log^k n$ we have the following aesthetically pleasing corollary.

**Corollary 3.** *For any integer $k \geq 1$,*

$$\sum_{n\leq x}{}' \frac{1}{\pi_k(n)} \ll_k \frac{x}{\pi_{k+1}(x)}.$$

Note that this corollary can be interpreted as saying the average value of $1/\pi_k(n)$ on $[1, x]$ is bounded by a constant multiple of $1/\pi_{k+1}(x)$. Lastly, we give a similar theorem involving the von Mangoldt function $\Lambda_k(n) := \Lambda_{\mathbb{P}_k}(n)$.

**Theorem 4.** *For any integer $k \geq 1$ and a fixed integer $N > k$ the function $\Lambda_k(n)$ satisfies*

$$\sum_{\substack{n\leq x \\ n\in\mathbb{N}_k}}{}' \frac{1}{\Lambda_k(n)} = \sum_{j=k+1}^{N} \frac{c_j x}{\log^j x} + O_k\left(\frac{x \log\log x}{\log^{k+2} x}\right), \tag{5}$$

*where $c_{k+1} = 1$ and the remaining $c_j$ are computable constants.*

The proof relies on the following bound for the generalized Chebyshev theta function $\theta_k(x) := \theta_{\mathbb{P}_k}(x)$.

**Lemma 5.** *For any integer $k \geq 1$ and a fixed positive integer $N$ the function $\theta_k(x)$ satisfies*

$$\theta_k(x) = \frac{x}{\log^{k-1} x} - \sum_{j=k}^{N} \frac{(j-1)!}{(k-1)!} \frac{x}{\log^j x} + O_k\left(\frac{x \log\log x}{\log^k x}\right), \tag{6}$$

*where the summation is understood to be zero if $N < k$.*

Both proofs of Theorem 4 and Lemma 5 are given below in Section 4.

## 2. Proof of Theorem 1

The basis of the theorem is the following bound for the $n$-th prime.

**Lemma 6.** *The function $\pi_{-1}(n)$ satisfies*

$$\frac{\pi_{-1}(n)}{n} = \log n + \log\log n - 1 + O\left(\frac{\log\log n}{\log n}\right).$$

*Proof.* This follows from a bound of Pervouchine which can be found in [Cesàro 1894]. □

Next, we give a lemma that will be used to bound compositions of $\pi_{-1}(n)$.

**Lemma 7.** *Let $f(n, j)$ be defined as*

$$f(n, j) := \log n + j \log\log n - 1.$$

*Then for any fixed integers $j, l \geq 1$ the following three equations hold*:

$$\log(nf(n, j)) = \log n + \log\log n + O_j\left(\frac{\log\log n}{\log n}\right), \tag{7}$$

$$\log\log(nf(n, j)) = \log\log n + O_j\left(\frac{\log\log n}{\log n}\right), \tag{8}$$

$$f(nf(n, j), l) = f(n, l+1) + O_{j,l}\left(\frac{\log\log n}{\log n}\right). \tag{9}$$

*Proof.* The proof is a straightforward computation, but we include it here for completeness. First, using the definition of $f(n, j)$ and properties of logarithms, we have

$$\begin{aligned}\log(nf(n, j)) &= \log n + \log(\log n + j\log\log n - 1)\\ &= \log n + \log\left(\log n\left(1 + j\frac{\log\log n}{\log n} - \frac{1}{\log n}\right)\right)\\ &= \log n + \log\log n + \log\left(1 + j\frac{\log\log n}{\log n} - \frac{1}{\log n}\right).\end{aligned}$$

Now (7) follows since

$$\log\left(1 + j\frac{\log\log n}{\log n} - \frac{1}{\log n}\right) \ll_j \frac{\log\log n}{\log n}$$

by expanding the logarithm. Similarly, from (7) we obtain

$$\begin{aligned}\log\log(nf(n, j)) &= \log\left(\log n + \log\log n + O_j\left(\frac{\log\log n}{\log n}\right)\right)\\ &= \log\left(\log n\left(1 + \frac{\log\log n}{\log n} + O_j\left(\frac{\log\log n}{\log^2 n}\right)\right)\right)\\ &= \log\log n + \log\left(1 + \frac{\log\log n}{\log n} + O_j\left(\frac{\log\log n}{\log^2 n}\right)\right),\end{aligned}$$

and (8) follows from expanding the last term. Finally, (9) follows immediately from the definition of $f$ and (7) and (8):

$$\begin{aligned}f(nf(n, j), l) &= \log(nf(n, j)) + l\log\log(nf(n, j)) - 1\\ &= \log n + \log\log n + O_j\left(\frac{\log\log n}{\log n}\right) + l\left(\log\log n + O_j\left(\frac{\log\log n}{\log n}\right)\right) - 1\\ &= \log n + (l+1)\log\log n - 1 + O_{j,l}\left(\frac{\log\log n}{\log n}\right).\end{aligned}$$

□

**Remark.** The three equations in Lemma 7 remain valid if we replace $f(n, j)$ by $f(n, j) + O(\log\log n/\log n)$ as the computations remain unchanged.

*Proof of Theorem 1.* We proceed via induction on $k$. Since the $k = 1$ case reduces to Lemma 6, suppose (3) holds for some $k = m > 1$; that is,

$$\pi_{-m}(n) = n \prod_{j=1}^{m} f(n, j) + O_m(n \log^{m-2} n \log\log n).$$

Recall that $\pi_{-(m+1)} = \pi_{-m} \circ \pi_{-1}$, so we have

$$\pi_{-(m+1)}(n) \\ = \pi_{-1}(n) \prod_{j=1}^{m} f(\pi_{-1}(n), j) + O_m(\pi_{-1}(n) \log^{m-2}(\pi_{-1}(n)) \log\log(\pi_{-1}(n))).$$

Note that by Lemmas 6 and 7

$$f(\pi_{-1}(n), l) = f\left(nf(n, 1) + O\left(\frac{n \log\log n}{\log n}\right), l\right) = f(n, l+1) + O_l\left(\frac{\log\log n}{\log n}\right)$$

for any $l \geq 1$. Applying this and Lemma 6 to the expression for $\pi_{-(m+1)}(n)$ above yields

$$\pi_{-(m+1)}(n) = n \prod_{j=1}^{m+1} \left(f(n, j) + O_j\left(\frac{\log\log n}{\log n}\right)\right) \\ + O_m\big(nf(n, 1) \log^{m-2}(nf(n, 1)) \log\log(nf(n, 1))\big).$$

The error terms arising from the product are at most

$$n \log^m n \, O_m\left(\frac{\log\log n}{\log n}\right) = O_m(n \log^{m-1} n \log\log n),$$

and by Lemma 7 we have

$$nf(n, 1) \log^{m-2}(nf(n, 1)) \log\log(nf(n, 1)) \ll n \log^{m-1} n \log\log n.$$

Thus,

$$\pi_{-(m+1)}(n) = n \prod_{j=1}^{m+1} f(n, j) + O_m(n \log^{m-1} n \log\log n),$$

which completes the proof. □

## 3. Proof of Theorem 2

The proof depends on the following two lemmas.

**Lemma 8.** *For any integer $k \geq 1$ the function $\pi_k(n)$ satisfies*

$$\pi_k(n) = \frac{n}{\log^k n} + O_k\left(\frac{n \log\log n}{\log^{k+1} n}\right). \tag{10}$$

*Proof.* This follows from the PNT; see Theorem 7.1 in [Bayless et al. 2013]. □

**Lemma 9.** *For any integer $k \geq 1$ we have*

$$\sum_{n \leq x} \frac{\log^k n}{n} = \frac{\log^{k+1} x}{k+1} + O_k(1). \tag{11}$$

*Proof.* Recall Abel's summation formula[2]

$$\sum_{y<n\leq x} a_n \phi(n) = A(x)\phi(x) - A(y)\phi(y) - \int_y^x A(t)\phi'(t)\,dt, \tag{12}$$

where $\phi \in \mathcal{C}^1$ and $A(x) = \sum_{n\leq x} a_n$ for some sequence $(a_n)$. Taking $a_n \equiv 1$ we have $A(x) = \lfloor x \rfloor = x - \{x\}$, where $\{x\}$ denotes the fractional part of $x$, and so

$$\sum_{y<n\leq x} \phi(n) = (x - \{x\})\phi(x) - (y - \{y\})\phi(y) - \int_y^x (t - \{t\})\phi'(t)\,dt.$$

Observing that $\int_y^x t\phi'(t)\,dt = x\phi(x) - y\phi(y) - \int_y^x \phi(t)\,dt$, we arrive at the following form of Abel's summation formula:

$$\sum_{y<n\leq x} \phi(n) = \int_y^x \phi(t)\,dt - \{x\}\phi(x) + \{y\}\phi(y) + \int_y^x \{t\}\phi'(t)\,dt.$$

Setting $y = 1$ and $\phi(x) = \log^k x/x$ yields

$$\sum_{n\leq x} \frac{\log^k n}{n} = \frac{\log^{k+1} x}{k+1} - \{x\}\frac{\log^k x}{x} + \int_1^x \{t\}\phi'(t)\,dt,$$

or, in other words,

$$\sum_{n\leq x} \frac{\log^k n}{n} - \frac{\log^{k+1} x}{k+1} \ll \phi(x) + \int_1^x |\phi'(t)|\,dt. \tag{13}$$

Now $\phi'(x) > 0$ on the interval $(1, e^k)$, and $\phi'(x) < 0$ on $(e^k, \infty)$. Thus, by the fundamental theorem of calculus, the right-hand side of (13) becomes

$$\phi(x) + \phi(e^k) - \phi(1) - \phi(x) + \phi(e^k) = 2\phi(e^k) = 2\left(\frac{k}{e}\right)^k,$$

which proves the lemma. □

[2] Abel's summation formula is integration by parts of a Riemann–Stieltjes integral; see Theorem 4.2 in [Apostol 1976] and Appendix A in [Montgomery and Vaughan 2007]. Several of the computations here, especially those in Section 4 below, become more straightforward with this machinery.

**Remark.** Lemma 9 also follows from the fact that

$$\lim_{x\to\infty}\left(\sum_{n\le x}\frac{\log^k n}{n}-\frac{\log^{k+1} x}{k+1}\right)=\gamma_k,$$

where $\gamma_k$ are constants, called Stieltjes constants, that occur in the Laurent series expansion of the Riemann zeta function about $s=1$; in particular,

$$\zeta(s)=\frac{1}{s-1}+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\gamma_k(s-1)^k;$$

see Example 1.3.12 on p. 29 of [Montgomery and Vaughan 2007]. Consequently, the proof of Lemma 9 also gives a simple upper bound for $|\gamma_k|$.

*Proof of Theorem 2.* The $k=1$ case is Theorem 9.1 in [De Koninck and Ivić 1980], so assume $k\ge 2$. From Lemmas 8 and 9 we have

$$\sum_{n\le x}{}'\left(\frac{1}{\pi_k(n)}-\frac{1}{n/\log^k n}\right)=O_k\left(\sum_{n\le x}{}'\frac{\log\log n\log^{k-1} n}{n}\right).$$

The theorem now follows from the fact that

$$\begin{aligned}\sum_{n\le x}{}'\frac{\log\log n\log^{k-1} n}{n}&\ll\int_1^x\frac{\log\log t\log^{k-1} t}{t}\,dt\\&=\log\log t\,\frac{\log^k t}{k}\bigg|_1^x-\frac{1}{k}\int_1^x\frac{\log^{k-1} t}{t}\,dt\\&=\frac{1}{k}\log\log x\log^k x-\frac{\log^k x}{k^2}.\end{aligned}$$

□

## 4. Proofs of Theorem 4 and Lemma 5

We first prove the bound for the Chebyshev theta function over $\mathbb{P}_k$.

*Proof of Lemma 5.* Using Riemann–Stieltjes integration, or Abel's summation formula in (12) with $\phi(x)=\log x$ and $A(x)=\pi_k(x)$, we have

$$\theta_k(x)=\int_2^x\log x\,d\pi_k(x)=\pi_k(x)\log x-\int_2^x\frac{\pi_k(t)}{t}\,dt$$

(note this reduces to Theorem 4.3 in [Apostol 1976] when $k=1$). By Lemma 8 this becomes

$$\theta_k(x)=\frac{x}{\log^{k-1} x}+O_k\left(\frac{x\log\log x}{\log^k x}\right)-\int_2^x\frac{dt}{\log^k t}-O_k\left(\int_2^x\frac{\log\log t}{\log^{k+1} t}\,dt\right).\qquad(14)$$

Next, integration by parts yields the formula

$$\int \frac{dt}{\log^k t} = \frac{t}{\log^k t} + k \int \frac{dt}{\log^{k+1} t},$$

which through repeated use gives

$$\int_2^x \frac{dt}{\log^k t} = \sum_{j=k}^{N} \frac{(j-1)!}{(k-1)!} \frac{x}{\log^j x} + O_k\left(\frac{x}{\log^{N+1} x}\right). \tag{15}$$

Similarly, using integration by parts on the integral in the error term, we have

$$\int \frac{\log\log t}{\log^{k+1} t}\, dt = \frac{t \log\log t}{\log^{k+1} t} - \int \frac{dt}{\log^{k+2} t} + (k+1) \int \frac{\log\log t}{\log^{k+2} t}\, dt,$$

which implies

$$\int_2^x \frac{\log\log t}{\log^{k+1} t}\, dt \ll \frac{x \log\log x}{\log^{k+1} x} \ll \frac{x \log\log x}{\log^k x}. \tag{16}$$

The lemma now follows from using (16) and (15) in (14). □

*Proof of Theorem 4*. The $k = 1$ case is Theorem 9.2 in [De Koninck and Ivić 1980], so assume $k \geq 2$. The idea of the proof is to use the definition of $\Lambda_k$ to write

$$\sideset{}{'}\sum_{\substack{n \leq x \\ n \in \mathbb{N}_k}} \frac{1}{\Lambda_k(n)} = \sum_{\substack{p^\alpha \leq x \\ \alpha \geq 1 \\ p \in \mathbb{P}_k}} \frac{1}{\log p} = \sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} + \sum_{\substack{p^2 \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} + \cdots$$

and then bound the summations on the right-hand side.[3] To this end, we again use Riemann–Stieltjes integration, or Abel's summation formula in (12) with $\phi(x) = 1/\log^2 x$ and $A(x) = \theta_k(x)$, to obtain

$$\sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} = \int_2^x \frac{1}{\log^2 t}\, d\theta_k(t) = \frac{\theta_k(x)}{\log^2 x} + 2 \int_2^x \frac{\theta_k(t)}{t \log^3 t}\, dt,$$

which by Lemma 5 becomes

$$\begin{aligned}\sum_{\substack{p \leq x \\ p \in \mathbb{P}_k}} \frac{1}{\log p} &= \frac{x}{\log^{k+1} x} - \sum_{j=k}^{N-2} \frac{(j-1)!}{(k-1)!} \frac{x}{\log^{j+2} x} + O_k\left(\frac{x \log\log x}{\log^{k+2} x}\right) \\ &\quad + 2 \int_2^x \frac{dt}{\log^{k+2} t} - 2 \sum_{j=k}^{N-3} \frac{(j-1)!}{(k-1)!} \int_2^x \frac{dt}{\log^{j+3} t} + O_k\left(\int_2^x \frac{\log\log t}{\log^{k+3} t}\, dt\right).\end{aligned} \tag{17}$$

[3]Note that there are only finitely many summations for a given $x$ because $p^\alpha \leq x$ implies $\alpha \leq \log x / \log p \leq \log x / \log 2$.

Note that the upper bounds on the summations have been chosen so that they terminate on the order of $x/\log^N x$. We can combine error terms by taking $k \to k+2$ in (16); that is,

$$\int_2^x \frac{\log\log t}{\log^{k+3} t}\,dt \ll \frac{x\log\log x}{\log^{k+3} x} \ll \frac{x\log\log x}{\log^{k+2} x}. \tag{18}$$

Now by (15) we have

$$\begin{aligned}\int_2^x \frac{dt}{\log^{k+2} t} &= \sum_{j=k+2}^{N} \frac{(j-1)!}{(k+1)!}\frac{x}{\log^j x} + O_k\left(\frac{x}{\log^{N+1} x}\right)\\ &= \sum_{j=k}^{N-2} \frac{(j+1)!}{(k+1)!}\frac{x}{\log^{j+2} x} + O_k\left(\frac{x}{\log^{N+1} x}\right),\end{aligned} \tag{19}$$

as well as

$$\int_2^x \frac{dt}{\log^{j+3} t} = \frac{x}{\log^{j+3} x} + O_j\left(\frac{x}{\log^{j+4} x}\right). \tag{20}$$

Putting (18)–(20) into (17) yields

$$\begin{aligned}\sum_{\substack{p\le x\\ p\in\mathbb{P}_k}} \frac{1}{\log p} = \frac{x}{\log^{k+1} x} &+ \sum_{j=k}^{N-2}\left(2\frac{(j+1)!}{(k+1)!} - \frac{(j-1)!}{(k-1)!}\right)\frac{x}{\log^{j+2} x}\\ &- 2\sum_{j=k}^{N-3}\frac{(j-1)!}{(k-1)!}\frac{x}{\log^{j+3} x} + O_k\left(\frac{x\log\log x}{\log^{k+2} x}\right),\end{aligned}$$

which can be rewritten as

$$\sum_{\substack{p\le x\\ p\in\mathbb{P}_k}} \frac{1}{\log p} = \sum_{j=k+1}^{N} \frac{c_j x}{\log^j x} + O_k\left(\frac{x\log\log x}{\log^{k+2} x}\right).$$

The theorem now follows from the fact that

$$\sum_{\substack{n\le x\\ n\in\mathbb{N}_k}}{}' \frac{1}{\Lambda_k(n)} = \sum_{\substack{p^\alpha\le x\\ \alpha\ge 1\\ p\in\mathbb{P}_k}} \frac{1}{\log p} = \sum_{\substack{p\le x\\ p\in\mathbb{P}_k}} \frac{1}{\log p} + O\left(\sum_{\substack{p^2\le x\\ p\in\mathbb{P}_k}} \frac{1}{\log p}\right),$$

and by observing that

$$\sum_{\substack{p\le\sqrt{x}\\ p\in\mathbb{P}_k}} \frac{1}{\log p} \ll \frac{\sqrt{x}}{\log^k \sqrt{x}} \ll \frac{x\log\log x}{\log^{k+2} x}$$

by Lemma 8. □

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