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Covering numbers and schlicht functions

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We determine upper and lower bounds for the minimal number of balls of a given radius needed to cover the space of schlicht functions.

1. Introduction

Let (X, d) be a metric space. The *δ-covering number* of X, denoted by $N_X(\delta)$, is the minimal number of closed balls of radius δ needed to cover X. It may happen that $N_X(\delta) = \infty$, but if X is compact then always $N_X(\delta) < \infty$. The quantity $N_X(\delta)$ arises in connection with the notion of the *box-counting dimension* of X, which is defined as

$$\dim_B(X) := \lim_{\delta \to 0^+} \frac{\log N_X(\delta)}{\log(1/\delta)},$$

whenever the limit exists; see, e.g., [Falconer 2014, §2.1]. The covering number also plays an important role in the theory of universal approximation; see, e.g., [Kalmes et al. 2012]. In the latter context, the metric space X is usually a function space.

Our aim is to estimate the covering number for a particular function space, namely the class $\mathcal{S}$ of *schlicht functions*. This is the class of all holomorphic functions f on the open unit disk $\mathbb{D}$ such that f is injective and satisfies $f(0) = 0$ and $f'(0) = 1$. The class $\mathcal{S}$ arises naturally in connection with the Riemann mapping theorem. It has been studied extensively and is the subject of several books, e.g., [Duren 1983; Goodman 1983a 1983b; Pommerenke 1975; Schober 1975].

2. Statement of main result

Let $\text{Hol}(\mathbb{D})$ denote the set of all holomorphic functions on the open unit disk $\mathbb{D}$. It is a Fréchet space with respect to the topology of uniform convergence on compact

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subsets of $\mathbb{D}$. There are many choices of metric that give rise to this topology. A very common one is to take

$$d(f, g) := \sum_{j\geq 1} \lambda_j \min\bigl\{1, \max_{|z|\leq r_j} |f(z) - g(z)|\bigr\}, \tag{1}$$

where (λ_j) is a positive sequence with $\sum_j \lambda_j < \infty$, and r_j is a sequence in $(0, 1)$ such that $r_j \to 1$.

With respect to this metric, the set $\mathcal{S}$ is a compact subset of $\mathrm{Hol}(\mathbb{D})$, see, e.g., [Duren 1983, p.276], and therefore a compact metric space in its own right. Our aim is to estimate the covering number of this space. In order to do this, we need to impose some conditions on the sequences (λ_j) and (r_j). We shall assume that there exist constants $0 < \lambda < 1$ and $\alpha > 0$ such that

$$\lambda_j \asymp \lambda^j \quad \text{and} \quad 1 - r_j \asymp j^{-\alpha}. \tag{2}$$

These choices seem reasonable. In the literature, λ_j is nearly always taken to be 2^{-j}. Usually r_j is not specified explicitly, but in [Duren 1983, §9.1] it is taken to be $1 - 1/j$.

Here and in what follows, we write $a \lesssim b$ to signify that a, b are positive functions or sequences such that a/b remains bounded above and $a \asymp b$ to signify that $a \lesssim b$ and $b \lesssim a$ hold together.

The following theorem is our main result.

Theorem 2.1. *Let d be the metric on $\mathcal{S}$ given by (1), where (λ_j) and (r_j) satisfy (2). Then*

$$\log^2(1/\delta) \lesssim \log N_{\mathcal{S}}(\delta) \lesssim \log^{2+\alpha}(1/\delta), \quad \delta \to 0^+. \tag{3}$$

The lower bound in (3) is a generalization of a result obtained in [Kalmes et al. 2012]. It shows, in particular, that the box-counting dimension of $\mathcal{S}$ is infinite. We believe that the upper bound in (3) is new.

3. Proof of the lower bound in Theorem 2.1

Our starting point is the following sufficient condition for membership in $\mathcal{S}$. The result is well known, but we include a short proof for convenience.

Proposition 3.1. *Let $f(z) := z + \sum_{k\geq 2} a_k z^k$, where $\sum_{k\geq 2} k|a_k| \leq 1$. Then $f \in \mathcal{S}$.*

Proof. We may suppose that at least one a_k is nonzero, otherwise the result is obvious. Let z, w be distinct points of $\mathbb{D}$. Then, by the triangle inequality,

$$|f(z) - f(w)| \geq |z - w| - \Bigl|\sum_{k\geq 2} a_k z^k - \sum_{k\geq 2} w^k\Bigr|.$$

Now

$$\left|\sum_{k\geq 2} a_k z^k - \sum_{k\geq 2} a_k w^k\right| \leq \sum_{k\geq 2} |a_k||z^k - w^k| < \sum_{k\geq 2} k|a_k||z-w| \leq |z-w|,$$

the middle inequality being strict because at least one a_k is nonzero. It follows that $f(z) \neq f(w)$. Thus f is injective, and so $f \in \mathcal{S}$. □

Let $\mathcal{A}$ be the family of functions $f(z) := z + \sum_{k\geq 2} a_k z^k$ such that $\sum_{k\geq 2} k|a_k| \leq 1$. By virtue of Proposition 3.1, we have $\mathcal{A} \subset \mathcal{S}$, and therefore $N_{\mathcal{S}}(\delta) \geq N_{\mathcal{A}}(2\delta)$. Thus, to establish the lower bound in Theorem 2.1, it suffices to prove the following theorem.

Theorem 3.2. *Let d be the metric on $\mathcal{A}$ given by (1), where (λ_j) and (r_j) satisfy (2). Then*

$$\log N_{\mathcal{A}}(\delta) \gtrsim \log^2(1/\delta), \quad \delta \to 0^+. \tag{4}$$

To prove this result, it is first convenient to establish a lemma.

Lemma 3.3. *Let $f, g \in \mathcal{A}$, say $f(z) := z + \sum_{k\geq 2} a_k z^k$ and $g(z) := z + \sum_{k\geq 2} b_k z^k$. Then*

$$d(f, g) \geq \sum_{k\geq 2} (\lambda_k r_k^k / 2)|a_k - b_k|.$$

Proof. The standard Cauchy estimate for Taylor coefficients gives

$$|a_k - b_k| \leq \max_{|z|\leq r_k} \frac{|f(z) - g(z)|}{r^k}.$$

Also, since $|a_k|, |b_k|, r_k \leq 1$, we have $|a_k - b_k| r_k^k / 2 \leq 1$. Combining these observations, we obtain

$$\begin{aligned} d(f, g) &= \sum_{k\geq 1} \lambda_k \min\{1, \max_{|z|\leq r_k} |f(z) - g(z)|\} \\ &\geq \sum_{k\geq 2} \lambda_k \min\{1, |a_k - b_k| r_k^k\} \geq \sum_{k\geq 2} \lambda_k |a_k - b_k| r_k^k / 2. \end{aligned}$$ □

Proof of Theorem 3.2. Let $n \geq 2$, and let K be a large integer depending on n, to be chosen later. Consider the collection of functions

$$f(z) := z + \sum_{k=2}^{n} \frac{1}{kn} \frac{t_k}{K} z^k, \quad t_k \in \{1, 2, \ldots, K\},\ k = 2, \ldots, n.$$

All of these functions belong to $\mathcal{A}$. There are K^{n-1} of them. Also, by Lemma 3.3, the d-distance between any two of them, f, g say, is at least

$$d(f, g) \geq \left(\min_{2\leq k\leq n} \frac{\lambda_k r_k^k}{2}\right) \frac{1}{n^2 K}.$$

No ball of radius one third the right-hand side can contain more than one of these functions. It follows that

$$N_{\mathcal{A}}\left(\frac{1}{3}\left(\min_{2\le k\le n}\frac{\lambda_k r_k^k}{2}\right)\frac{1}{n^2K}\right)\ge K^{n-1}. \tag{5}$$

By our assumptions, there exists $\alpha>0$ such that $1-r_k\asymp k^{-\alpha}$. It follows that $r_k^k\ge e^{-ck^{1-\alpha}}$ for some constant $c>0$. Since also $\lambda_k\ge C\lambda^k$ for some constant $C>0$, we have

$$\frac{1}{3}\left(\min_{2\le k\le n}\frac{\lambda_k r_k^k}{3}\right)\frac{1}{n^2}\ge\frac{C}{6}\exp(-n\log(1/\lambda)-cn^{1-\alpha}-2\log n).$$

If we choose $\rho\in(0,\lambda)$, then $n\log(1/\lambda)+cn^{1-\alpha}+2\log n\le n\log(1/\rho)$ for all sufficiently large n, and for these n we then have

$$\left(\min_{2\le k\le n}\frac{\lambda_k r_k^k}{6}\right)\frac{1}{n^2}\ge\rho^n. \tag{6}$$

Reducing ρ, if necessary, we can ensure that this inequality holds for all n, and also that $1/\rho$ is an integer. Note that ρ may depend on the sequences (λ_k) and (r_k), but it is independent of K. Combining the inequalities (5) and (6), we obtain

$$N_{\mathcal{A}}(\rho^n/K)\ge K^{n-1}.$$

We now choose K to be $K:=\rho^{-n}$. This gives

$$N_{\mathcal{A}}(\rho^{2n})\ge\rho^{-n(n-1)}.$$

As this holds for each $n\ge 2$, we deduce that (4) holds. □

4. Proof of the upper bound in Theorem 2.1

This time, our starting point is the famous result of [de Branges 1985] that proved the Bieberbach conjecture.

Theorem 4.1. *Let $f\in\mathcal{S}$, say $f(z)=\sum_{k\ge0}a_kz^k$. Then $|a_k|\le k$ for all k.*

Special cases of this result had previously been established by various mathematicians for small values of k. Much earlier, Littlewood [1925] had proved the slightly weaker (but much easier) estimate $|a_k|\le ek$ valid for all k. In fact Littlewood's estimate would do for our purposes.

Again, it is convenient to introduce some notation. We denote by $\mathcal{B}$ the family of functions $f(z):=\sum_{k\ge1}a_kz^k$ such that $|a_k|\le k$ for all k. By virtue of Theorem 4.1, we have $\mathcal{S}\subset\mathcal{B}$, and so $N_{\mathcal{S}}(\delta)\le N_{\mathcal{B}}(\delta/2)$. Thus, to establish the upper bound in Theorem 2.1, it suffices to prove the following theorem.

Theorem 4.2. *Let d be the metric on $\mathcal{B}$ given by (1), where (λ_j) and (r_j) satisfy (2). Then*

$$\log N_{\mathcal{B}}(\delta) \lesssim \log^{2+\alpha}(1/\delta), \quad \delta \to 0^+. \tag{7}$$

For the proof, we need the following approximation lemma.

Lemma 4.3. *There exists a constant $c > 0$ with the following property: for each $f \in \mathcal{B}$ and $n \geq 1$, there is a polynomial $p \in \mathcal{B}$ such that* $\deg p \leq n$ *and*

$$d(f, p) \leq \exp(-cn^{1/(1+\alpha)}).$$

Proof. Let $f \in \mathcal{B}$, say $f(z) = \sum_{k\geq 1} a_k z^k$. Let $n \geq 1$ and set $p(z) := \sum_{k=1}^n a_k z^k$. Clearly $p \in \mathcal{B}$ and $\deg p \leq n$. Also

$$d(f, p) \leq \sum_{j\geq 1} \lambda_j \max_{|z|\leq r_j} \left| \sum_{k\geq n+1} a_k z^k \right| \leq \sum_{j\geq 1} \left(\lambda_j \sum_{k\geq n+1} k r_j^k \right).$$

So, to prove the lemma, it suffices to show that

$$\sum_{j\geq 1} \left(\lambda_j \sum_{k\geq n+1} k r_j^k \right) \leq \exp(-cn^{1/(1+\alpha)}), \quad n \geq 1, \tag{8}$$

where $c > 0$ is a constant independent of n.

Now, for $r \in (0, 1)$, we have

$$\sum_{k\geq n+1} k r^k = r \frac{d}{dr} \left(\sum_{k\geq n+1} r^k \right) = r \frac{d}{dr} \left(\frac{r^{n+1}}{1-r} \right) \leq (n+2) \frac{r^{n+1}}{(1-r)^2}.$$

Therefore

$$\sum_{j\geq 1} \left(\lambda_j \sum_{k\geq n+1} k r_j^k \right) \leq \sum_{j\geq 1} \lambda_j (n+2) \frac{r_j^{n+1}}{(1-r_j)^2}.$$

Recalling that $\lambda_j \asymp \lambda^j$ and $1 - r_j \asymp j^{-\alpha}$, we see that there are constants $C_1, c_1 > 0$, independent of n, such that

$$\sum_{j\geq 1} \lambda_j (n+2) \frac{r_j^{n+1}}{(1-r_j)^2} \leq C_1 n \sum_{j\geq 1} \lambda^j j^{2\alpha} \exp(-c_1 n j^{-\alpha}).$$

Now

$$\begin{aligned} \sum_{j\leq n^{1/(1+\alpha)}} \lambda^j j^{2\alpha} \exp(-c_1 n j^{-\alpha}) &\leq \sum_{j\leq n^{1/(1+\alpha)}} \lambda^j n^{2\alpha/(1+\alpha)} \exp(-c_1 n^{1/(1+\alpha)}) \\ &\leq \frac{1}{1-\lambda} n^{2\alpha/(1+\alpha)} \exp(-c_1 n^{1/(1+\alpha)}). \end{aligned}$$

Also

$$\sum_{j>n^{1/(1+\alpha)}} \lambda^j j^{2\alpha} \exp(-c_1 n j^{-\alpha}) \leq \sum_{j>n^{1/(1+\alpha)}} \lambda^j j^{2\alpha}$$

$$\leq \big(\sup_{j\geq 1} \lambda^{j/2} j^{2\alpha}\big) \sum_{j>n^{1/(1+\alpha)}} \lambda^{j/2} \leq C_2 \frac{(\sqrt{\lambda})^{n^{1/(1+\alpha)}}}{1-\sqrt{\lambda}},$$

where C_2 is another constant. Combining all these inequalities, we see that (8) holds for any positive constant $c < \min\{c_1, \log(1/\sqrt{\lambda})\}$, at least for all sufficiently large n. Reducing c further, if necessary, we can ensure that (8) holds for all n. □

Remark. The estimate in Lemma 4.3 is sharp. Indeed, consider the so-called Koebe function, namely $f(z) := z/(1-z)^2 = \sum_{k\geq 1} kz^k$. This belongs to $\mathcal{B}$, and even to $\mathcal{S}$. Also, for every polynomial p of degree n, whether it lies in $\mathcal{B}$ or not, we have

$$\max_{|z|\leq r_j} |f(z)-p(z)|^2 \geq \frac{1}{2\pi r_j}\int_{|z|=r_j} |f(z)-p(z)|^2\,|dz| \geq \sum_{k\geq n+1} k^2 r_j^{2k} \geq r_j^{2n+2}.$$

It follows that $d(f,p) \geq \sum_{j\geq 1} \lambda_j r_j^{n+1}$. Retaining just the term with $j = \lfloor n^{1/(1+\alpha)} \rfloor$, we see that

$$d(f,p) \geq \exp(-cn^{1/(1+\alpha)}),$$

where $c > 0$ is a constant independent of p and n.

Proof of Theorem 4.2. Let $n \geq 1$, and let K be a large integer depending on n, to be chosen later. Consider the collection of polynomials

$$q(z) := \sum_{k=1}^{n} k\frac{(s_k+it_k)}{\sqrt{2}K} z^k, \quad s_k, t_k \in \{-K, -(K-1), \dots, K\},\ k = 1, \dots, n.$$

Clearly these polynomials all belong to $\mathcal{B}$, and there are $(2K+1)^{2n}$ of them. Given an arbitrary polynomial $p \in \mathcal{B}$ with $\deg p \leq n$, there is a member q of this collection whose coefficients all lie within n/K of the corresponding coefficients of p, and consequently

$$d(p,q) = \sum_{j\geq 1} \lambda_j \min\big\{1, \max_{|z|\leq r_j} |p(z)-q(z)|\big\} \leq \sum_{j\geq 1} \lambda_j \frac{n^2}{K} = Cn^2/K,$$

where C is a constant independent of n. Combining this inequality with the result of Lemma 4.3, we see that, if we set $\delta := Cn^2/K + \exp(-cn^{1/(1+\alpha)})$, then the δ-balls around the polynomials q cover $\mathcal{B}$. Hence

$$N_{\mathcal{B}}(Cn^2/K + \exp(-cn^{1/(1+\alpha)})) \leq (2K+1)^{2n}.$$

We now choose K to be $K := \lceil Cn^2 \exp(cn^{1/(1+\alpha)})\rceil$. This gives

$$\begin{aligned} N_{\mathcal{B}}(2\exp(-cn^{1/(1+\alpha)})) &\le (2\lceil Cn^2\exp(cn^{1/(1+\alpha)})\rceil+1)^{2n} \\ &= \exp(2n\log(2\lceil Cn^2\exp(cn^{1/(1+\alpha)})\rceil+1)) \\ &\le \exp(2n(\log(n^2)+cn^{1/(1+\alpha)}+O(1))) \\ &\le \exp(2cn^{(2+\alpha)/(1+\alpha)}+O(n\log n)). \end{aligned}$$

From this it is straightforward to deduce that (7) holds. □

5. Concluding remarks and questions

(1) Among several well-known subclasses of $\mathcal{S}$, there is the class $\mathcal{C}$ of convex functions, namely the set of those $f \in \mathcal{S}$ such that $f(\mathbb{D})$ is a convex set. Theorem 2.1 holds with $\mathcal{S}$ replaced by $\mathcal{C}$. Indeed, the upper bound is obvious. As for the lower bound, it suffices to repeat the same proof, using the following result of [Goodman 1957] in place of Lemma 3.3: if $f(z) := z + \sum_{k\ge 2} a_k z^k$, where $\sum_{k\ge 2} k^2|a_k| \le 1$, then $f \in \mathcal{C}$.

(2) Which, if either, of the bounds in (3) is sharp? The bounds are based on the inclusions $\mathcal{A} \subset \mathcal{S} \subset \mathcal{B}$. The spaces $\mathcal{A}$ and $\mathcal{B}$ are easier to handle than $\mathcal{S}$ because they resemble infinite cartesian products. However, to answer the question above, we probably need to use finer properties of $\mathcal{S}$ to determine which of $\mathcal{A}$ or $\mathcal{B}$ better approximates $\mathcal{S}$.

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