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Sharp sectional curvature bounds and a new proof of the spectral theorem

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We algebraically compute all possible sectional curvature values for canonical algebraic curvature tensors and use this result to give a method for constructing general sectional curvature bounds. We use a well-known method to geometrically realize these results to produce a hypersurface with prescribed sectional curvatures at a point. By extending our methods, we give a relatively short proof of the spectral theorem for self-adjoint operators on a finite-dimensional real vector space.

1. Introduction

Let V be a real vector space of finite dimension n, and let $V^* = \mathrm{Hom}(V, \mathbb{R})$ be the corresponding dual space of $\mathbb{R}$-linear maps from V to $\mathbb{R}$. An *algebraic curvature tensor* $R \in \otimes^4 V^*$ satisfies the properties below for all $x, y, z, w \in V$:

$$\begin{aligned} R(x, y, z, w) &= -R(y, x, z, w) = R(z, w, x, y), \\ R(x, y, z, w) &+ R(x, w, y, z) + R(x, z, w, y) = 0. \end{aligned} \tag{1.a}$$

Let $\mathcal{A}(V)$ denote the set of all algebraic curvature tensors. Let $S^2(V^*)$ denote the space of all symmetric bilinear forms over V, and let $\varphi \in S^2(V^*)$. Define a *canonical algebraic curvature tensor* $R_\varphi \in \mathcal{A}(V)$ as

$$R_\varphi(x, y, z, w) = \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w).$$

If the vector space V is endowed with a nondegenerate inner product $\langle \cdot, \cdot \rangle$, then the triple $(V, \langle \cdot, \cdot \rangle, R)$ is referred to as a *model space*. In this research, all inner products are assumed to be positive definite. Throughout this work, we consider the vector space V and a positive definite inner product $\langle \cdot, \cdot \rangle$ to be given, and so we do not always specifically reference them. For the sake of convenience, we will refer to properties of a model space (such as sectional curvature defined below) as a property of an algebraic curvature tensor when there is no possibility of confusion.

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Let $\mathrm{Gr}_2(V)$ be the set of all 2-planes through the origin in V; this topological space is known as the *Grassmannian* of 2-planes in V. This space is compact and connected [Milnor and Stasheff 1974]. Given a model space $(V, \langle \cdot, \cdot \rangle, R)$ and a 2-plane $\pi \in V$, define the *sectional curvature* $\kappa(\pi)$ of π to be

$$\frac{R(x, y, y, x)}{\langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2},$$

where $\pi = \mathrm{span}\{x, y\}$. This quantity is independent of the basis chosen for π.

Interest in algebraic curvature tensors stems from basic results in differential geometry. If (M, g) is a smooth manifold, one may compute the Riemann curvature tensor R_P at the point $P \in M$ using the Levi-Civita connection. It is well known that R_P satisfies the identities listed in (1.a), and thus R_P is an algebraic curvature tensor on the tangent space $T_P M$. Using the metric g_P restricted to $T_P M$, $(T_P M, g_P, R_P)$ is a model space. The converse is also true [Gilkey 2007]: given a model space $(V, \langle \cdot, \cdot \rangle, R)$, there exists a smooth manifold (M, g), a point $P \in M$, and a vector space isometry $\Psi : V \to T_P M$ so that $\Psi^* R_P = R$. The process of finding such a manifold is generally referred to as a geometric realization.

A general line of questioning is to explore exactly what one can say about a manifold that is a geometric realization of any given model space(s). In this way, the algebraic properties of the model space can influence the geometry of a geometric realization of it. Although there are many examples of this [Brozos Vázquez et al. 2012; García-Río et al. 2002; Gilkey and Nikčević 2005; Kowalski and Prüfer 1994; Tricerri and Vanhecke 1981; 1986; Tsankov 2005], we give two examples relevant to our study. The first example concerns the property of constant sectional curvature: up to local isometry, there is only one way to construct a manifold such that the model space at any of its points has constant sectional curvature.

The second example begins with the fact [Gilkey 2007] that $\{R_\varphi \mid \varphi \in S^2(V^*)\}$ is spanning set for the space of algebraic curvature tensors. For $R \in \mathcal{A}(V)$, define

$$\nu(R) = \min\Big\{k \;\Big|\; R = \textstyle\sum_{i=1}^k \alpha_i R_{\varphi_i}\Big\}, \quad \nu(n) = \max\{\nu(R) \mid R \in \mathcal{A}(V)\}.$$

This notation seems to have originated in [Gilkey 2007] and has since been studied by several authors. Díaz-Ramos and García-Río [2004] showed that $\nu(3) = 2$ and used the Nash embedding theorem [1956] to prove $\nu(n) \leq n(n+1)/2$. In this way, $\nu(R)$ functions as a lower bound of the codimension of any local isometric embedding of a given manifold, where R is the algebraic curvature tensor at a given point. For this reason, subsequent work [Diaz and Dunn 2010] has aimed to discover linear dependencies in the set of canonical algebraic curvature tensors.

We have two major goals for this research. First, we establish sharp bounds on the sectional curvature values of any canonical algebraic curvature tensor. We interpret these bounds through a geometric realization result, and give a method for constructing bounds on the sectional curvature values of any algebraic curvature tensor. By an extension of our methods (inspired by R. Klinger [1991]), we meet our second goal: to give a short and self-contained proof of the spectral theorem.

More specifically, after some preliminary comments, in Section 2 we establish:

Theorem 1.1. *Let $\varphi \in S^2(V^*)$, and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of φ, repeated according to multiplicity. Let m and M, respectively, be the minimum and maximum of the set $\{\lambda_i\lambda_j \mid i \neq j\}$. The set of sectional curvatures of R_φ is precisely the interval $[m, M]$.*

We use this result to prove two corollaries. The first corollary (Corollary 2.3) uses a well-known result to geometrically realize any interval as the set of sectional curvatures of a manifold at a point. The manifolds we produce are hypersurfaces in Euclidean space. The second corollary (Corollary 2.4) provides bounds (which are not sharp, see Remarks 2.5 and 2.6) on the set of sectional curvatures of an arbitrary algebraic curvature tensor R in terms of $\nu(R)$ and the results from Theorem 1.1.

In Section 3 we consider a canonical algebraic curvature tensor to provide a short and self-contained proof of the spectral theorem:

Theorem 1.2 (the spectral theorem). *Let V be an inner product space. If $\varphi \in S^2(V^*)$, then there exists an orthonormal basis $\{f_1, \dots, f_n\}$ for V for which $\varphi(f_i, f_j) = \lambda_i\delta_{ij}$.*

2. Sectional curvature bounds

It is well known that given $\varphi \in S^2(V^*)$, there exists a self-adjoint linear map $A : V \to V$ characterized by the equation $\varphi(x, y) = \langle Ax, y\rangle$. This follows as an application of the Riesz representation theorem (see page 188 of [Axler 2015]), and we briefly pause to describe this correspondence.

If a linear map $A : V \to V$ is given, then one can define the bilinear form $\varphi(x, y) = \langle Ax, y\rangle$ for vectors $x, y \in V$. If, in addition, A is self-adjoint with respect to the inner product (i.e., $\langle Ax, y\rangle = \langle x, Ay\rangle$), then φ is symmetric. Conversely, if $\varphi \in S^2(V^*)$ is given, one could construct the (unique) self-adjoint linear map A satisfying $\varphi(x, y) = \langle Ax, y\rangle$ in the following way. First, choose an orthonormal basis $\mathcal{B} = \{f_1, \dots, f_n\}$ for V. Then compute the numbers $a_{ji} = \langle Af_i, f_j\rangle$. The matrix for A relative to this orthonormal basis is $[A]_\mathcal{B} = [a_{ij}]$.

Because of the bijective correspondence between symmetric bilinear forms and their corresponding self-adjoint linear maps, relevant aspects of φ are defined in terms of those same aspects in A, for example, $\ker(\varphi)$ and $\mathrm{Rank}(\varphi)$ are defined as $\ker(A)$ and $\mathrm{Rank}(A)$, respectively.

Similarly, eigenvalues and eigenvectors of a linear map $A : V \to V$ can related to the corresponding bilinear form φ. It will be helpful to do so in reference to an orthonormal basis: as described above, if $\{f_1, \dots, f_n\}$ is an orthonormal basis for V, the (j, i) matrix entry of A on this basis is equal to $\varphi(f_i, f_j)$. For this reason, f_i is an eigenvector with corresponding eigenvalue λ_i if and only if $\varphi(f_i, f_j) = \lambda_i \delta_{ij}$.

It is through this perspective that we establish our results. We will repeatedly use the following lemma; our proof uses a technique adapted from [Klinger 1991]. Translated into our terminology, the objective in that paper is to find a basis (called a Chern basis) for V that minimizes the number of nonzero entries of a given (arbitrary) algebraic curvature tensor. This is accomplished by first considering a 2-plane π with extremal sectional curvature, and then rotating this plane around an axis (within the plane) by an angle of θ to create the rotated plane π_θ, with $\pi_0 = \pi$. The sectional curvature function has an extremal value at $\theta = 0$, and so this is a critical point of the function $\theta \mapsto \kappa(\pi_\theta)$. Using the fact that the derivative of this function at $\theta = 0$ vanishes, one can uncover information about the components of the algebraic curvature tensor. We have adapted this technique (which seems to be an extension of the work in [Bishop and Goldberg 1964]) to apply to the algebraic curvature tensor R_φ to diagonalize the symmetric bilinear form φ.

Lemma 2.1. *Let $\varphi \in S^2(V^*)$, and suppose $R_\varphi \neq 0$. If π is a 2-plane whose sectional curvature is extremal, then there exists an orthonormal basis of eigenvectors for π.*

Proof. Since $R_\varphi \neq 0$, and R_φ is determined by its sectional curvatures [Lee 1997], there exists a 2-plane π of extremal nonzero sectional curvature. Let $\{e_1, e_2\}$ be any orthonormal basis for π. Let $\varphi|_\pi$ be the restriction of φ to π. We now diagonalize[1] the symmetric form $\varphi|_\pi$ to create a new orthonormal basis $\{f_1, f_2\}$ for π. For some θ, set

$$f_1 = \cos\theta e_1 - \sin\theta e_2,$$
$$f_2 = \sin\theta e_1 + \cos\theta e_2,$$

where we will determine θ presently. Let $\varphi_{ij} = \varphi(e_i, e_j)$. We compute

$$\begin{aligned}\varphi(f_1, f_2) &= \cos\theta\sin\theta(\varphi_{11} - \varphi_{22}) + (\cos^2\theta - \sin^2\theta)\varphi_{12}\\ &= \tfrac{1}{2}\sin(2\theta)(\varphi_{11} - \varphi_{22}) + \cos(2\theta)\varphi_{12}.\end{aligned}$$

Choose θ so that $\varphi(f_1, f_2) = 0$. Explicitly, if $\varphi_{12} = 0$ already, then $\theta = 0$. Otherwise,

$$\theta = \tfrac{1}{2}\operatorname{arccot}\left(\frac{\varphi_{22} - \varphi_{11}}{2\varphi_{12}}\right).$$

Now extend this basis for π (in an arbitrary way) to form an orthonormal basis $\{f_1, f_2, f_3, \ldots, f_n\}$ for V. Note that on this basis,

$$\begin{aligned}&\varphi(f_1, f_2) = 0,\\ &R_\varphi(f_1, f_2, f_2, f_1) = \varphi(f_1, f_1)\varphi(f_2, f_2) \quad \text{is extremal and nonzero.}\end{aligned} \tag{2.a}$$

[1]One could of course use the spectral theorem here, but since we use this result in our proof of the spectral theorem in the next section, we instead establish this directly.

We now show that $\varphi(f_1, f_j) = \varphi(f_2, f_j) = 0$ for $j \geq 3$, which will complete the proof. Choose any index $j \geq 3$, and consider the plane

$$\pi_\theta = \text{span}\{\cos\theta f_1 + \sin\theta f_j, f_2\}.$$

Using a double angle formula and various curvature identities listed in the Introduction,

$$\kappa(\pi_\theta) = \cos^2\theta R_\varphi(f_1, f_2, f_2, f_1) + \sin^2\theta R_\varphi(f_j, f_2, f_2, f_j) + \sin(2\theta) R_\varphi(f_1, f_2, f_2, f_j).$$

Since π_0 is extremal, 0 is a critical point of the function $\theta \mapsto \kappa(\pi_\theta)$, so

$$0 = \frac{d}{d\theta}[\kappa(\pi_\theta)]|_{\theta=0} = 2R_\varphi(f_1, f_2, f_2, f_j).$$

As a result,

$$\begin{aligned} 0 = R_\varphi(f_1, f_2, f_2, f_j) &= \varphi(f_1, f_j)\varphi(f_2, f_2) - \varphi(f_1, f_2)\varphi(f_2, f_j) \\ &= \varphi(f_1, f_j)\varphi(f_2, f_2). \end{aligned}$$

By (2.a), we know $\varphi(f_2, f_2) \neq 0$; hence $\varphi(f_1, f_j) = 0$. We can prove that $\varphi(f_2, f_j) = 0$ in a similar way by considering $\text{span}\{f_1, \cos\theta f_2 + \sin\theta f_j\}$. □

Remark 2.2. Lemma 2.1 relates to Euler's theorem, which states (among other things) that at any point on a 2-dimensional manifold, the so-called principal directions are orthogonal and are eigenvectors of a certain bilinear form (provided these eigenvalues are different). This establishes Lemma 2.1 in dimension 2: the curvature tensor of a 2-dimensional manifold is of the form R_φ, where φ is this bilinear form.

We use Lemma 2.1 to establish Theorem 1.1.

Proof of Theorem 1.1. We start by proving that the maximal sectional curvature is M, the largest pairwise product of eigenvalues of φ. Since $\text{Gr}_2(V)$ is compact, there exists a 2-plane Π for which

$$\kappa(\Pi) = \sup\{\kappa(L) \mid L \in \text{Gr}_2(V)\}.$$

We proceed in cases: either $\kappa(\Pi) \neq 0$ or $\kappa(\Pi) = 0$. Our goal is to show $\kappa(\Pi)$ to be a pairwise product of eigenvalues. Thus, as the maximal sectional curvature value, $\kappa(\Pi) = M$.

If $\kappa(\Pi) \neq 0$, then we may use Lemma 2.1 to produce an orthonormal basis of eigenvectors $\{f_1, f_2\}$ for Π. In this case,

$$\kappa(\Pi) = R_\varphi(f_1, f_2, f_2, f_1) = \varphi(f_1, f_1)\varphi(f_2, f_2)$$

is a product of eigenvalues and hence is equal to M.

Now suppose $\kappa(\Pi) = 0$. Find an orthonormal basis $\{f_1, \dots, f_n\}$ diagonalizing φ, and note

$$\kappa(\text{span}\{f_i, f_j\}) = R_\varphi(f_i, f_j, f_j, f_i) = \varphi(f_i, f_i)\varphi(f_j, f_j)$$

is a product of eigenvalues of φ. It is not possible that there are two nonzero eigenvalues of the same sign since their corresponding eigenvectors would span a 2-plane of positive sectional curvature which is contrary to our assumption that $\kappa(\Pi) = 0$ is the maximal sectional curvature. Since $\dim(V) \geq 3$, we know 0 is an eigenvalue, and therefore $0 = \kappa(\Pi) = M$ is the largest pairwise product of eigenvalues as desired.

We now prove that the minimal sectional curvature is m, the smallest pairwise product of eigenvalues of φ. Proceeding similarly, there exists a 2-plane π for which

$$\kappa(\pi) = \inf\{\kappa(L) \mid L \in \text{Gr}_2(V)\}.$$

We again proceed in cases: either $\kappa(\pi) \neq 0$ or $\kappa(\pi) = 0$. If $\kappa(\pi) \neq 0$ then use Lemma 2.1 once more to produce an orthonormal basis of eigenvectors $\{f_1, f_2\}$ for π. Then

$$\kappa(\pi) = R_\varphi(f_1, f_2, f_2, f_1) = \varphi(f_1, f_1)\varphi(f_2, f_2)$$

is a product of eigenvalues and is hence equal to m.

Now suppose $\kappa(\pi) = 0$ and again diagonalize $\varphi|_\pi$ with the basis $\{f_1, f_2\}$ and extend it to an orthonormal basis $\{f_1, \dots, f_n\}$ for V. We then have

$$0 = \kappa(\pi) = R_\varphi(f_1, f_2, f_2, f_1) = \varphi(f_1, f_1)\varphi(f_2, f_2).$$

Thus, one of the two factors above is zero. As exchanging the two vectors would not change the span or disrupt the diagonalization, we may assume $\varphi(f_1, f_1) = 0$. For $j \geq 3$, since 0 is assumed to be the minimal sectional curvature, we find

$$\begin{aligned} 0 \leq \kappa(\text{span}\{f_1, f_j\}) &= R_\varphi(f_1, f_j, f_j, f_1) \\ &= \varphi(f_1, f_1)\varphi(f_j, f_j) - \varphi(f_1, f_j)^2 \\ &= -\varphi(f_1, f_j)^2. \end{aligned}$$

It follows that $\varphi(f_1, f_j) = 0$ for all $j \neq 1$, and so f_1 is an eigenvector corresponding to the eigenvalue $\varphi(f_1, f_1) = 0$. Thus, 0 is an eigenvalue of φ.

To finish this portion of the proof, we prove that all other eigenvalues of φ are either zero or have the same sign, which would show 0 to be the smallest pairwise product of the eigenvalues. To do this, find an orthonormal basis $\{e_1, \dots, e_n\}$ for V that diagonalizes φ. Since 0 is minimal, we must have

$$0 \leq \kappa(\text{span}\{e_i, e_j\}) = R_\varphi(e_i, e_j, e_j, e_i) = \varphi(e_i, e_i)\varphi(e_j, e_j).$$

For this reason the nonzero eigenvalues of φ cannot differ in sign, and this portion of the proof is complete.

We have proven that $\kappa(\Pi) = M$ and $\kappa(\pi) = m$ are, respectively, the maximal and minimal sectional curvatures. Since $\mathrm{Gr}_2(V)$ is connected and the mapping $L \mapsto \kappa(L)$ is continuous, the image of this mapping is connected as well, which completes the proof. □

We use a well-known geometric realization result to establish the following corollary.

Corollary 2.3. *Let* $[a, b]$ *be any interval. There exists a hypersurface* M *in Euclidean space, a smooth metric* g *on* M, *and a point* $P \in M$ *so that the set of sectional curvatures of* M *at* P *is precisely* $[a, b]$.

Proof. Choose any collection of real numbers $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ (not necessarily distinct) whose minimal and maximal pairwise products are, respectively, a and b. Define the symmetric bilinear form φ so that $\lambda_1, \dots, \lambda_n$ are its eigenvalues. One may now carry out the geometric realization procedure as outlined on page 74 of [Gilkey 2001] to produce the desired result. □

Before proving our next corollary, we note that $R_{c\varphi} = c^2 R_\varphi$, and so any linear combination of canonical algebraic curvature tensors satisfies

$$\sum \alpha_i R_{\varphi_i} = \sum \epsilon_i R_{\tilde{\varphi}_i},$$

where $\epsilon_i = \pm 1 = \mathrm{sign}(\alpha_i)$ and $\tilde{\varphi}_i = \sqrt{|\alpha_i|}\varphi_i$.

The following corollary finds bounds on the sectional curvature values of any algebraic curvature tensor and is a direct consequence of Theorem 1.1.

Corollary 2.4. *Let* $R \in \mathcal{A}(V)$, *and set* $R = \sum_{i=1}^{\nu(R)} \epsilon_i R_{\varphi_i}$, *where* $\epsilon_i = \pm 1$ *and* $\varphi_i \in S^2(V^*)$. *Let* M_i *and* m_i *be, respectively, the maximal and minimal pairwise products of the eigenvalues of* φ_i, *and let* $M = \sum \epsilon_i M_i$ *and* $m = \sum \epsilon_i m_i$. *The set of possible sectional curvature values of the model space* $(V, \langle \cdot, \cdot \rangle, R)$ *is a closed subinterval of* $[m, M]$.

Remark 2.5. We give an example which shows that the bounds presented in Corollary 2.4 are not sharp. On an orthonormal basis $\{f_1, f_2, f_3\}$, define $\varphi_1, \varphi_2 \in S^2(V^*)$ by having the nonzero entries

$$\varphi_1(f_1, f_1) = \varphi_1(f_2, f_2) = \varphi_2(f_1, f_1) = \varphi_2(f_3, f_3) = 1.$$

According to Theorem 1.1, the maximal sectional curvature of both R_{φ_1} and R_{φ_2} is 1, so Corollary 2.4 estimates the maximal sectional curvature of $R_{\varphi_1} + R_{\varphi_2}$ to be 2. A straightforward calculation shows, however, that all sectional curvatures of $R_{\varphi_1} + R_{\varphi_2}$ are strictly less than 2. □

Remark 2.6. The authors in [Bettiol and Mendes 2017] independently obtained general sectional curvature bounds. While their methods involve representation theory and differential geometric considerations, our approach provides a more basic proof (especially in the case of a canonical algebraic curvature tensor) and can be extended to give a basic proof of the spectral theorem, which we give in the next section.

3. The spectral theorem

In this section we give a relatively short proof of the well-known spectral theorem after establishing a helpful lemma.

Lemma 3.1. *Suppose φ is a symmetric bilinear form on an inner product space V with* $\dim(V) \geq 2$. *There exists an orthonormal basis $\{f_1, e_2, \ldots, e_n\}$ for V so that f_1 is an eigenvector of φ.*

Proof. Let φ be given, and consider the algebraic curvature tensor R_φ. We break the proof into cases: either $R_\varphi \neq 0$ or $R_\varphi = 0$.

Suppose first that $R_\varphi \neq 0$. Using Lemma 2.1, we choose a 2-plane π of extremal sectional curvature and find an orthonormal basis $\{f_1, e_2\}$ of eigenvectors for π, which we extend to the orthonormal basis $\{f_1, e_2, \ldots, e_n\}$ for V, establishing the result.

Now suppose $R_\varphi = 0$. We claim that 0 is an eigenvalue[2] of φ. In this case, any corresponding unit eigenvector may be chosen first as part of an orthonormal basis for V, which would again establish the result.

Suppose to the contrary that 0 is not an eigenvalue of φ. Therefore, for any orthonormal basis $\{f_1, \ldots, f_n\}$, the matrix with $\varphi(f_j, f_i)$ as its (i, j)-entry must have a nonzero determinant. By expanding this determinant by cofactors, there must be at least one nonzero 2×2 minor. Thus, for some indices i_1, i_2, j_1, j_2,

$$\begin{aligned} 0 &\neq \varphi(f_{i_1}, f_{j_1})\varphi(f_{i_2}, f_{j_2}) - \varphi(f_{i_1}, f_{j_2})\varphi(f_{i_2}, f_{j_1}) \\ &= R_\varphi(f_{i_1}, f_{i_2}, f_{j_2}, f_{j_1}), \end{aligned}$$

which contradicts the assumption that $R_\varphi = 0$. □

With these short preliminary facts established, we can now give an alternative proof of the spectral theorem.

Proof of the spectral theorem. The result is trivial if $\dim(V) = 1$, so assume $\dim(V) \geq 2$. By Lemma 3.1, there exists an orthonormal basis $\{f_1, e_2, \ldots, e_n\}$ for

[2]In fact, it is known [Gilkey 2007] that if $R_\varphi = 0$ then the rank of φ is less than or equal to 1, so that on a vector space of dimension 2 or more, there must be a nontrivial kernel of φ. However, we wish to give a self-contained and alternative proof here as part of our effort to produce a new proof of the spectral theorem.

V so that f_1 is an eigenvector of φ. Now consider the vector space $W = f_1^{\perp} = \operatorname{span}\{e_2, \dots, e_n\}$. Since our inner product is positive definite, the restriction of the inner product to W remains positive definite. In addition, the restricted curvature tensor satisfies $(R_\varphi)|_W = R_{\varphi|_W}$.

We first consider when $\dim(W) \geq 2$. We again use Lemma 3.1 to find an orthonormal basis $\{f_2, \tilde{e}_3, \dots, \tilde{e}_n\}$ for W so that f_2 is an eigenvector of $\varphi|_W$. That is,

$$\varphi|_W(f_2, \tilde{e}_k) = \varphi(f_2, \tilde{e}_k) = 0.$$

Since f_1 is an eigenvector of φ, we have $\varphi(f_1, f_2) = 0$, so f_2 is also an eigenvector of φ. We have a new orthonormal basis $\{f_1, f_2, \tilde{e}_3, \dots, \tilde{e}_n\}$ for V, where both f_1 and f_2 are eigenvectors of φ.

Now consider $W_2 = \operatorname{span}\{f_1, f_2\}^{\perp}$. If $\dim(W_2) > 1$, then repeat the process outlined above, and continue until there is an orthonormal basis $\{f_1, \dots, f_{n-1}, \bar{e}_n\}$ for V (this is the case if $\dim(W) = 1$ above), and for any $i \neq j$ inclusively between 1 and $n - 1$ we have

$$\varphi(f_i, f_j) = \varphi(f_i, \bar{e}_n) = 0.$$

This demonstrates that $f_n = \bar{e}_n$ is also an eigenvector, completing the proof. □

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