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Conjecture $\mathcal{O}$ holds for some horospherical varieties of Picard rank 1

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Property $\mathcal{O}$ for an arbitrary complex, Fano manifold X is a statement about the eigenvalues of the linear operator obtained from the quantum multiplication of the anticanonical class of X. Conjecture $\mathcal{O}$ is a conjecture that property $\mathcal{O}$ holds for any Fano variety. Pasquier classified the smooth nonhomogeneous horospherical varieties of Picard rank 1 into five classes. Conjecture $\mathcal{O}$ has already been shown to hold for the odd symplectic Grassmannians, which is one of these classes. We will show that conjecture $\mathcal{O}$ holds for two more classes and an example in a third class of Pasquier's list. Perron–Frobenius theory reduces our proofs to be graph-theoretic in nature.

1. Introduction

The purpose of this paper is to prove that conjecture $\mathcal{O}$ holds for some horospherical varieties of Picard rank 1. We recall the precise statement of conjecture $\mathcal{O}$ for varieties of Picard rank 1, following [Galkin et al. 2016, Section 3]. Let F be a Fano variety, let $K := K_F$ be the canonical line bundle of F, let F_D be a fundamental divisor of F, and let

$$c_1(F) := c_1(-K) \in H^2(F)$$

be the anticanonical class. The Fano index of F is r, where r is the greatest integer such that $K_F \cong -rF_D$. The small quantum cohomology ring $(QH^*(F), \star)$ is a graded algebra over $\mathbb{Z}[q]$, where q is the quantum parameter. We define the small quantum cohomology in Section 2.1. Consider the specialization

$$H^\bullet(F) := QH^*(F)|_{q=1}$$

at $q = 1$. The quantum multiplication by the first Chern class $c_1(F)$ induces an endomorphism $\hat{c}_1$ of the finite-dimensional vector space $H^\bullet(F)$:

$$y \in H^\bullet(F) \mapsto \hat{c}_1(y) := (c_1(F) \star y)|_{q=1}.$$

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Set $\delta_0 := \max\{|\delta| : \delta$ is an eigenvalue of $\hat{c}_1\}$. Then property $\mathcal{O}$ states the following:

(1) The real number δ_0 is an eigenvalue of $\hat{c}_1$ of multiplicity 1.

(2) If δ is any eigenvalue of $\hat{c}_1$ with $|\delta| = \delta_0$, then $\delta = \delta_0 \gamma$ for some r-th root of unity $\gamma \in \mathbb{C}$, where r is the Fano index of F.

Property $\mathcal{O}$ was conjectured to hold for any Fano, complex manifold F in [Galkin et al. 2016]. If a Fano, complex, manifold has property $\mathcal{O}$ then we say that the space satisfies conjecture $\mathcal{O}$. Conjecture $\mathcal{O}$ underlies gamma conjectures I and II of Galkin, Golyshev, and Iritani. The gamma conjectures refine earlier conjectures by Dubrovin on Frobenius manifolds and mirror symmetry. Conjecture $\mathcal{O}$ has already been proved for the homogeneous G/P case in [Cheong and Li 2017], the odd symplectic Grassmannians in [Li et al. 2019], del Pezzo surfaces in [Hu et al. 2019], and projective complete intersections in [Ke 2018]. The Perron–Frobenius theory of nonnegative matrices reduces the proofs that conjecture $\mathcal{O}$ holds for the homogeneous and the odd symplectic Grassmannian cases to be a graph-theoretic check. This is because conjecture $\mathcal{O}$ is largely reminiscent of Perron–Frobenius theory. In this manuscript we will use the same graph-theoretic approach to prove that conjecture $\mathcal{O}$ holds for some smooth horospherical varieties of Picard rank 1.

Next we recall the definition of a horospherical variety following [Gonzales et al. 2018]. Let G be a complex reductive group. A G-variety is a reduced scheme of finite type over the field of complex numbers $\mathbb{C}$, equipped with an algebraic action of G. Let B be a Borel subgroup of G. A G-variety X is called spherical if X has a dense B-orbit. Let X be a G-spherical variety and let H be the stabilizer of a point in the dense G-orbit in X. The variety X is called *horospherical* if H contains a conjugate of the maximal unipotent subgroup of G contained in the Borel subgroup B.

Smooth horospherical varieties of Picard rank 1 were classified in [Pasquier 2009]. These varieties are either homogeneous or can be constructed in a uniform way via a triple $(\mathrm{Type}(G), \omega_Y, \omega_Z)$ of representation-theoretic data, where $\mathrm{Type}(G)$ is the semisimple Lie type of the reductive group G and ω_Y, ω_Z are fundamental weights. See [Pasquier 2009, Section 1.3] for details. Pasquier classified the possible triples into five classes:

(1) $(B_n, \omega_{n-1}, \omega_n)$ with $n \geq 3$.

(2) $(B_3, \omega_1, \omega_3)$.

(3) $(C_n, \omega_m, \omega_{m-1})$ with $n \geq 2$ and $m \in [2, n]$ (the odd symplectic Grassmannians).

(4) $(F_4, \omega_2, \omega_3)$.

(5) $(G_2, \omega_1, \omega_2)$.

Proposition 3.6 [Pasquier 2006] showed the triples in the above list are Fano varieties. We are now able to state the main theorem:

Theorem 1. *If F belongs to the classes* (1) *for $n = 3$,* (2), (3), *and* (5) *of Pasquier's list, then conjecture $\mathcal{O}$ holds for F.*

2. Preliminaries

2.1. ***Quantum cohomology.*** The small quantum cohomology is defined as follows. Let $(\alpha_i)_i$ be a basis of $H^*(F)$, the classical cohomology ring, and let $(\alpha_i^\vee)_i$ be the dual basis for the Poincaré pairing. The multiplication is given by

$$\alpha_i \star \alpha_j = \sum_{d\geq 0,k} c_{i,j}^{k,d} q^d \alpha_k,$$

where $c_{i,j}^{k,d}$ are the 3-point, genus-0, Gromov–Witten invariants corresponding to the classes α_i, α_j, and $\alpha_k^\vee$. We will make use of the quantum Chevalley formula, which is the multiplication of a hyperplane class h with another class α_j. Theorem 0.0.3 of [Gonzales et al. 2018] implies that if F belongs to the classes (1) for $n = 3$, (2), or (5) of Pasquier's list, then there is an explicit quantum Chevalley formula. The explicit quantum Chevalley formula is the key ingredient used to prove property $\mathcal{O}$ holds.

2.2. ***Sufficient criterion for property $\mathcal{O}$ to hold.*** We recall the notion of the (oriented) quantum Bruhat graph of a Fano variety F. The vertices of this graph are the basis elements

$$\alpha_i \in H^\bullet(F) := QH^*(F)|_{q=1}.$$

There is an oriented edge $\alpha_i \to \alpha_j$ if the class α_j appears with positive coefficient (where we consider $q > 0$) in the quantum Chevalley multiplication $h \star \alpha_i$ for some hyperplane class h. Using the Perron–Frobenius theory of nonnegative matrices, conjecture $\mathcal{O}$ reduces to a graph-theoretic check of the quantum Bruhat graph. The techniques involving Perron–Frobenius theory used by Li, Mihalcea, and Shifler [Li et al. 2019] and Cheong and Li [2017] imply the following lemma:

Lemma 2. *Suppose the following conditions hold for a Fano variety F:*

(1) *The matrix representation of $\hat{c}_1$ is nonnegative.*

(2) *The quantum Bruhat graph of F is strongly connected.*

(3) *There exists a cycle of length r, the Fano index, in the quantum Bruhat graph of F.*

Then property $\mathcal{O}$ holds for F. We say the matrix representation of $\hat{c}_1$ is nonnegative if all of the entries are nonnegative.

We refer the reader to [Minc 1988, Section 4.3] for further details on Perron–Frobenius theory.

3. Checking property $\mathcal{O}$ holds

Let X be a horospherical variety. We will simplify our notation where the basis of $H^\bullet(X)$ is $\{1, h, \alpha_i\}_{i\in I}$ for some finite index set I. Observe by [Gonzales et al. 2018] that the anticanonical classes are

$$c_1(X) = \begin{cases} 5h & \text{when } X \text{ is case (1) for } n=3, \\ 7h & \text{when } X \text{ is case (2)}, \\ 4h & \text{when } X \text{ is case (5)}, \end{cases}$$

and the Fano indices are

$$r = \begin{cases} 5 & \text{when } X \text{ is case (1) for } n=3, \\ 7 & \text{when } X \text{ is case (2)}, \\ 4 & \text{when } X \text{ is case (5)}. \end{cases}$$

The endomorphism $\hat{c}_1$ acting on the basis elements of $H^\bullet(X)$ is determined by the Chevalley formula in the following way:

$$\begin{aligned} \hat{c}_1(\alpha_i) &= 5(h \star \alpha_i)|_{q=1} && \text{when } X \text{ is case (1) for } n=3, \\ \hat{c}_1(\alpha_i) &= 7(h \star \alpha_i)|_{q=1} && \text{when } X \text{ is case (2), and} \\ \hat{c}_1(\alpha_i) &= 4(h \star \alpha_i)|_{q=1} && \text{when } X \text{ is case (5).} \end{aligned}$$

Each of the following three subsections will show that conjecture $\mathcal{O}$ holds for case (1) for $n=3$, case (2), and case (5) of Pasquier's list, respectively. In each subsection we will reformulate the quantum Chevalley formulas stated in [Gonzales et al. 2018], present the quantum Bruhat graph, and argue that each condition of Lemma 2 is satisfied. For each case, we have kept the same format of the equations presented by [Gonzales et al. 2018] with our prescribed basis for ease of identification for the reader. For example, line 3 in Proposition 4.3 of that paper is

$$h * \sigma'_{u_2} = \sigma'_{u_3} + \sigma'_{u'_3} \quad \text{and} \quad h * \sigma'_{u'_2} = 2\sigma'_{u'_3} + \tau_{v_0}.$$

In Proposition 3 below we identify this line with

$$\hat{c}_1(\alpha_1) = 5\alpha_3 + 5\alpha_4 \quad \text{and} \quad \hat{c}_1(\alpha_2) = 10\alpha_3 + 5\alpha_5.$$

3.1. ***Case* (1) *for* $n=3$.** We will reformulate the quantum Chevalley formula stated in [Gonzales et al. 2018] using the basis $\{1, h, \alpha_1, \alpha_2, \dots, \alpha_{18}\}$.

Proposition 3. *The following equalities hold by* [Gonzales et al. 2018, Proposition 4.3]:

(1) $\hat{c}_1(1) = 5h$.

(2) $\hat{c}_1(h) = 10\alpha_1 + 5\alpha_2$.

(3) $\hat{c}_1(\alpha_1) = 5\alpha_3 + 5\alpha_4$ *and* $\hat{c}_1(\alpha_2) = 10\alpha_3 + 5\alpha_5$.

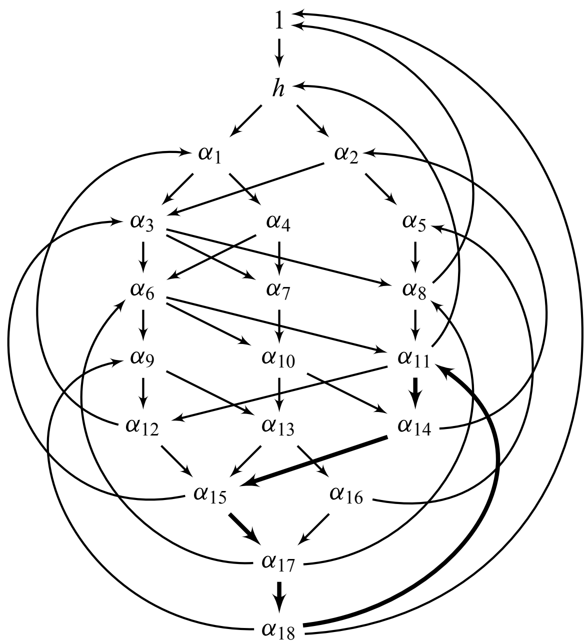

Figure 1. The quantum Bruhat graph of the Fano variety X in case (1) for $n = 3$.

(4) $\hat{c}_1(\alpha_3) = 10\alpha_6 + 5\alpha_7 + 5\alpha_8$, $\hat{c}_1(\alpha_4) = 5\alpha_6 + 10\alpha_7$, *and* $\hat{c}_1(\alpha_5) = 5\alpha_8$.

(5) $\hat{c}_1(\alpha_6) = 10\alpha_9 + 5\alpha_{10} + 5\alpha_{11}$, $\hat{c}_1(\alpha_7) = 5\alpha_{10}$, *and* $\hat{c}_1(\alpha_8) = 5\alpha_{11} + 5 \cdot 1$.

(6) $\hat{c}_1(\alpha_9) = 5\alpha_{12} + 5\alpha_{13}$, $\hat{c}_1(\alpha_{10}) = 10\alpha_{13} + 5\alpha_{14}$, $\hat{c}_1(\alpha_{11}) = 5\alpha_{12} + 5\alpha_{14} + 5h$.

(7) $\hat{c}_1(\alpha_{12}) = 5\alpha_{15} + 5\alpha_1$, $\hat{c}_1(\alpha_{13}) = 5\alpha_{15} + 5\alpha_{16}$, *and* $\hat{c}_1(\alpha_{14}) = 5\alpha_{15} + 5\alpha_2$.

(8) $\hat{c}_1(\alpha_{15}) = 5\alpha_{17} + 5\alpha_3$ *and* $\hat{c}_1(\alpha_{16}) = 5\alpha_{17} + 5\alpha_5$.

(9) $\hat{c}_1(\alpha_{17}) = 5\alpha_{18} + 5\alpha_6 + 5\alpha_8$.

(10) $\hat{c}_1(\alpha_{18}) = 5\alpha_9 + 5\alpha_{11} + 10 \cdot 1$.

The quantum Bruhat graph is shown in Figure 1. The bold edges indicate a cycle of length $r = 5$, the Fano index.

Lemma 4. *Property $\mathcal{O}$ holds when X is case* (1) *with $n = 3$ of Pasquier's list.*

Proof. The coefficients that appear in the equations in Proposition 3 are the entries of the matrix representation of $\hat{c}_1$. Therefore, the matrix representation of $\hat{c}_1$ is nonnegative. The quantum Bruhat graph is strongly connected by Figure 1, and the cycle $\alpha_{18}\alpha_{11}\alpha_{14}\alpha_{15}\alpha_{17}\alpha_{18}$ has length $r = 5$. □

3.2. ***Case* (2).** Again, we reformulate the quantum Chevalley formula from [Gonzales et al. 2018] using the basis $\{1, h, \alpha_1, \alpha_2, \ldots, \alpha_{12}\}$.

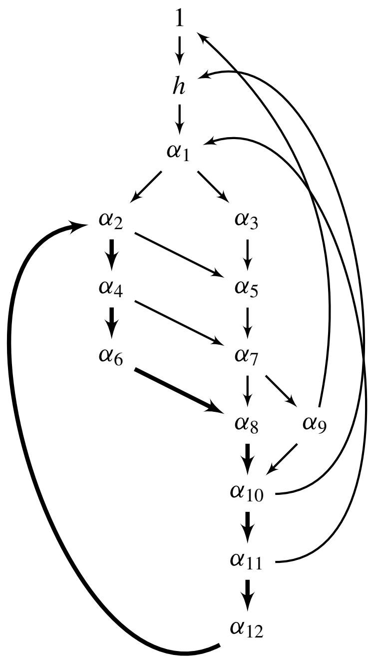

Figure 2. The quantum Bruhat graph for case (2).

Proposition 5. *The following equalities hold by* [Gonzales et al. 2018, Proposition 4.4]:

(1) $\hat{c}_1(1) = 7h$.

(2) $\hat{c}_1(h) = 7\alpha_1$.

(3) $\hat{c}_1(\alpha_1) = 14\alpha_2 + 7\alpha_3$.

(4) $\hat{c}_1(\alpha_2) = 7\alpha_4 + 7\alpha_5$ *and* $\hat{c}_1(\alpha_3) = 7\alpha_5$.

(5) $\hat{c}_1(\alpha_4) = 7\alpha_6 + 7\alpha_7$ *and* $\hat{c}_1(\alpha_5) = 7\alpha_7$.

(6) $\hat{c}_1(\alpha_6) = 7\alpha_8$ *and* $\hat{c}_1(\alpha_7) = 7\alpha_8 + 7\alpha_9$.

(7) $\hat{c}_1(\alpha_8) = 7\alpha_{10}$ *and* $\hat{c}_1(\alpha_9) = 7\alpha_{10} + 7 \cdot 1$.

(8) $\hat{c}_1(\alpha_{10}) = 7\alpha_{11} + 7h$.

(9) $\hat{c}_1(\alpha_{11}) = 7\alpha_{12} + 7\alpha_1$.

(10) $\hat{c}_1(\alpha_{12}) = 7\alpha_2$.

The quantum Bruhat graph is shown in Figure 2.

Lemma 6. *Property $\mathcal{O}$ holds when X is case* (2) *of Pasquier's list.*

Proof. The coefficients that appear in the equations in Proposition 5 are the entries of the matrix representation of $\hat{c}_1$. Therefore, the matrix representation of $\hat{c}_1$ is

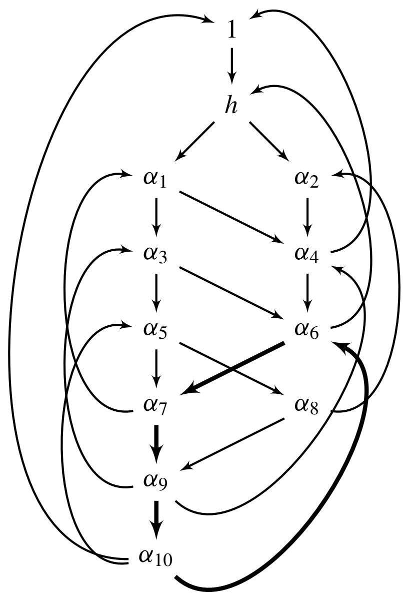

Figure 3. The quantum Bruhat graph for case (5).

nonnegative. The quantum Bruhat graph is strongly connected by Figure 2, and the cycle $\alpha_{12}\alpha_2\alpha_4\alpha_6\alpha_8\alpha_{10}\alpha_{11}\alpha_{12}$ has length $r = 7$. □

3.3. ***Case* (5).** Again, we reformulate the quantum Chevalley formula from [Gonzales et al. 2018] using the basis $\{1, h, \alpha_1, \alpha_2, \ldots, \alpha_{10}\}$.

Proposition 7. *The following equalities hold by* [Gonzales et al. 2018, Proposition 4.6]:

(1) $\hat{c}_1(1) = 4h$.

(2) $\hat{c}_1(h) = 12\alpha_1 + 4\alpha_2$.

(3) $\hat{c}_1(\alpha_1) = 8\alpha_3 + 4\alpha_4$ *and* $\hat{c}_1(\alpha_2) = 4\alpha_4$.

(4) $\hat{c}_1(\alpha_3) = 12\alpha_5 + 4\alpha_6$ *and* $\hat{c}_1(\alpha_4) = 4\alpha_6 + 4 \cdot 1$.

(5) $\hat{c}_1(\alpha_5) = 4\alpha_7 + 4\alpha_8$ *and* $\hat{c}_1(\alpha_6) = 8\alpha_7 + 4h$.

(6) $\hat{c}_1(\alpha_7) = 4\alpha_9 + 4\alpha_1$ *and* $\hat{c}_1(\alpha_8) = 4\alpha_9 + 4\alpha_2$.

(7) $\hat{c}_1(\alpha_9) = 4\alpha_{10} + 4\alpha_3 + 4\alpha_4$.

(8) $\hat{c}_1(\alpha_{10}) = 4\alpha_5 + 4\alpha_6 + 8 \cdot 1$.

The associated quantum Bruhat graph is shown in Figure 3.

Lemma 8. *Property* $\mathcal{O}$ *holds when* X *is case* (5) *of Pasquier's list.*

Proof. The coefficients that appear in the equations in Proposition 7 are the entries of the matrix representation of $\hat{c}_1$. Therefore, the matrix representation of $\hat{c}_1$ is nonnegative. The quantum Bruhat graph is strongly connected by Figure 3, and the cycle $\alpha_{10}\alpha_6\alpha_7\alpha_9\alpha_{10}$ has length $r = 4$. □

Theorem 1 follows from Lemmas 4, 6, 8, and the previously mentioned work done by Li, Mihalcea, Shifler [Li et al. 2019] for the odd symplectic Grassmannian case.

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