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Conjecture  $\mathcal{O}$  holds for some horospherical varieties of Picard  
rank 1

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# Conjecture $\mathcal{O}$ holds for some horospherical varieties of Picard rank 1

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Property  $\mathcal{O}$  for an arbitrary complex, Fano manifold  $X$  is a statement about the eigenvalues of the linear operator obtained from the quantum multiplication of the anticanonical class of  $X$ . Conjecture  $\mathcal{O}$  is a conjecture that property  $\mathcal{O}$  holds for any Fano variety. Pasquier classified the smooth nonhomogeneous horospherical varieties of Picard rank 1 into five classes. Conjecture  $\mathcal{O}$  has already been shown to hold for the odd symplectic Grassmannians, which is one of these classes. We will show that conjecture  $\mathcal{O}$  holds for two more classes and an example in a third class of Pasquier’s list. Perron–Frobenius theory reduces our proofs to be graph-theoretic in nature.

## 1. Introduction

The purpose of this paper is to prove that conjecture  $\mathcal{O}$  holds for some horospherical varieties of Picard rank 1. We recall the precise statement of conjecture  $\mathcal{O}$  for varieties of Picard rank 1, following [Galkin et al. 2016, Section 3]. Let  $F$  be a Fano variety, let  $K := K_F$  be the canonical line bundle of  $F$ , let  $F_D$  be a fundamental divisor of  $F$ , and let

$$c_1(F) := c_1(-K) \in H^2(F)$$

be the anticanonical class. The Fano index of  $F$  is  $r$ , where  $r$  is the greatest integer such that  $K_F \cong -rF_D$ . The small quantum cohomology ring  $(QH^*(F), \star)$  is a graded algebra over  $\mathbb{Z}[q]$ , where  $q$  is the quantum parameter. We define the small quantum cohomology in Section 2.1. Consider the specialization

$$H^\bullet(F) := QH^*(F)|_{q=1}$$

at  $q = 1$ . The quantum multiplication by the first Chern class  $c_1(F)$  induces an endomorphism  $\hat{c}_1$  of the finite-dimensional vector space  $H^\bullet(F)$ :

$$y \in H^\bullet(F) \mapsto \hat{c}_1(y) := (c_1(F) \star y)|_{q=1}.$$

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*Keywords:* quantum cohomology, horospherical, conjecture  $\mathcal{O}$ .

Set  $\delta_0 := \max\{|\delta| : \delta \text{ is an eigenvalue of } \hat{c}_1\}$ . Then property  $\mathcal{O}$  states the following:

- (1) The real number  $\delta_0$  is an eigenvalue of  $\hat{c}_1$  of multiplicity 1.
- (2) If  $\delta$  is any eigenvalue of  $\hat{c}_1$  with  $|\delta| = \delta_0$ , then  $\delta = \delta_0\gamma$  for some  $r$ -th root of unity  $\gamma \in \mathbb{C}$ , where  $r$  is the Fano index of  $F$ .

Property  $\mathcal{O}$  was conjectured to hold for any Fano, complex manifold  $F$  in [Galkin et al. 2016]. If a Fano, complex, manifold has property  $\mathcal{O}$  then we say that the space satisfies conjecture  $\mathcal{O}$ . Conjecture  $\mathcal{O}$  underlies gamma conjectures I and II of Galkin, Golyshev, and Iritani. The gamma conjectures refine earlier conjectures by Dubrovin on Frobenius manifolds and mirror symmetry. Conjecture  $\mathcal{O}$  has already been proved for the homogeneous  $G/P$  case in [Cheong and Li 2017], the odd symplectic Grassmannians in [Li et al. 2019], del Pezzo surfaces in [Hu et al. 2019], and projective complete intersections in [Ke 2018]. The Perron–Frobenius theory of nonnegative matrices reduces the proofs that conjecture  $\mathcal{O}$  holds for the homogeneous and the odd symplectic Grassmannian cases to be a graph-theoretic check. This is because conjecture  $\mathcal{O}$  is largely reminiscent of Perron–Frobenius theory. In this manuscript we will use the same graph-theoretic approach to prove that conjecture  $\mathcal{O}$  holds for some smooth horospherical varieties of Picard rank 1.

Next we recall the definition of a horospherical variety following [Gonzales et al. 2018]. Let  $G$  be a complex reductive group. A  $G$ -variety is a reduced scheme of finite type over the field of complex numbers  $\mathbb{C}$ , equipped with an algebraic action of  $G$ . Let  $B$  be a Borel subgroup of  $G$ . A  $G$ -variety  $X$  is called spherical if  $X$  has a dense  $B$ -orbit. Let  $X$  be a  $G$ -spherical variety and let  $H$  be the stabilizer of a point in the dense  $G$ -orbit in  $X$ . The variety  $X$  is called *horospherical* if  $H$  contains a conjugate of the maximal unipotent subgroup of  $G$  contained in the Borel subgroup  $B$ .

Smooth horospherical varieties of Picard rank 1 were classified in [Pasquier 2009]. These varieties are either homogeneous or can be constructed in a uniform way via a triple  $(\text{Type}(G), \omega_Y, \omega_Z)$  of representation-theoretic data, where  $\text{Type}(G)$  is the semisimple Lie type of the reductive group  $G$  and  $\omega_Y, \omega_Z$  are fundamental weights. See [Pasquier 2009, Section 1.3] for details. Pasquier classified the possible triples into five classes:

- (1)  $(B_n, \omega_{n-1}, \omega_n)$  with  $n \geq 3$ .
- (2)  $(B_3, \omega_1, \omega_3)$ .
- (3)  $(C_n, \omega_m, \omega_{m-1})$  with  $n \geq 2$  and  $m \in [2, n]$  (the odd symplectic Grassmannians).
- (4)  $(F_4, \omega_2, \omega_3)$ .
- (5)  $(G_2, \omega_1, \omega_2)$ .

Proposition 3.6 [Pasquier 2006] showed the triples in the above list are Fano varieties. We are now able to state the main theorem:

**Theorem 1.** *If  $F$  belongs to the classes (1) for  $n = 3$ , (2), (3), and (5) of Pasquier’s list, then conjecture  $\mathcal{O}$  holds for  $F$ .*

## 2. Preliminaries

**2.1. Quantum cohomology.** The small quantum cohomology is defined as follows. Let  $(\alpha_i)_i$  be a basis of  $H^*(F)$ , the classical cohomology ring, and let  $(\alpha_i^\vee)_i$  be the dual basis for the Poincaré pairing. The multiplication is given by

$$\alpha_i \star \alpha_j = \sum_{d \geq 0, k} c_{i,j}^{k,d} q^d \alpha_k,$$

where  $c_{i,j}^{k,d}$  are the 3-point, genus-0, Gromov–Witten invariants corresponding to the classes  $\alpha_i$ ,  $\alpha_j$ , and  $\alpha_k^\vee$ . We will make use of the quantum Chevalley formula, which is the multiplication of a hyperplane class  $h$  with another class  $\alpha_j$ . Theorem 0.0.3 of [Gonzales et al. 2018] implies that if  $F$  belongs to the classes (1) for  $n = 3$ , (2), or (5) of Pasquier’s list, then there is an explicit quantum Chevalley formula. The explicit quantum Chevalley formula is the key ingredient used to prove property  $\mathcal{O}$  holds.

**2.2. Sufficient criterion for property  $\mathcal{O}$  to hold.** We recall the notion of the (oriented) quantum Bruhat graph of a Fano variety  $F$ . The vertices of this graph are the basis elements

$$\alpha_i \in H^\bullet(F) := QH^*(F)|_{q=1}.$$

There is an oriented edge  $\alpha_i \rightarrow \alpha_j$  if the class  $\alpha_j$  appears with positive coefficient (where we consider  $q > 0$ ) in the quantum Chevalley multiplication  $h \star \alpha_i$  for some hyperplane class  $h$ . Using the Perron–Frobenius theory of nonnegative matrices, conjecture  $\mathcal{O}$  reduces to a graph-theoretic check of the quantum Bruhat graph. The techniques involving Perron–Frobenius theory used by Li, Mihalcea, and Shifler [Li et al. 2019] and Cheong and Li [2017] imply the following lemma:

**Lemma 2.** *Suppose the following conditions hold for a Fano variety  $F$ :*

- (1) *The matrix representation of  $\hat{c}_1$  is nonnegative.*
- (2) *The quantum Bruhat graph of  $F$  is strongly connected.*
- (3) *There exists a cycle of length  $r$ , the Fano index, in the quantum Bruhat graph of  $F$ .*

*Then property  $\mathcal{O}$  holds for  $F$ . We say the matrix representation of  $\hat{c}_1$  is nonnegative if all of the entries are nonnegative.*

We refer the reader to [Minc 1988, Section 4.3] for further details on Perron–Frobenius theory.

### 3. Checking property $\mathcal{O}$ holds

Let  $X$  be a horospherical variety. We will simplify our notation where the basis of  $H^\bullet(X)$  is  $\{1, h, \alpha_i\}_{i \in I}$  for some finite index set  $I$ . Observe by [Gonzales et al. 2018] that the anticanonical classes are

$$c_1(X) = \begin{cases} 5h & \text{when } X \text{ is case (1) for } n = 3, \\ 7h & \text{when } X \text{ is case (2),} \\ 4h & \text{when } X \text{ is case (5),} \end{cases}$$

and the Fano indices are

$$r = \begin{cases} 5 & \text{when } X \text{ is case (1) for } n = 3, \\ 7 & \text{when } X \text{ is case (2),} \\ 4 & \text{when } X \text{ is case (5).} \end{cases}$$

The endomorphism  $\hat{c}_1$  acting on the basis elements of  $H^\bullet(X)$  is determined by the Chevalley formula in the following way:

$$\begin{aligned} \hat{c}_1(\alpha_i) &= 5(h \star \alpha_i)|_{q=1} && \text{when } X \text{ is case (1) for } n = 3, \\ \hat{c}_1(\alpha_i) &= 7(h \star \alpha_i)|_{q=1} && \text{when } X \text{ is case (2), and} \\ \hat{c}_1(\alpha_i) &= 4(h \star \alpha_i)|_{q=1} && \text{when } X \text{ is case (5).} \end{aligned}$$

Each of the following three subsections will show that conjecture  $\mathcal{O}$  holds for case (1) for  $n = 3$ , case (2), and case (5) of Pasquier’s list, respectively. In each subsection we will reformulate the quantum Chevalley formulas stated in [Gonzales et al. 2018], present the quantum Bruhat graph, and argue that each condition of Lemma 2 is satisfied. For each case, we have kept the same format of the equations presented by [Gonzales et al. 2018] with our prescribed basis for ease of identification for the reader. For example, line 3 in Proposition 4.3 of that paper is

$$h \star \sigma'_{u_2} = \sigma'_{u_3} + \sigma'_{u'_3} \quad \text{and} \quad h \star \sigma'_{u'_2} = 2\sigma'_{u'_3} + \tau_{v_0}.$$

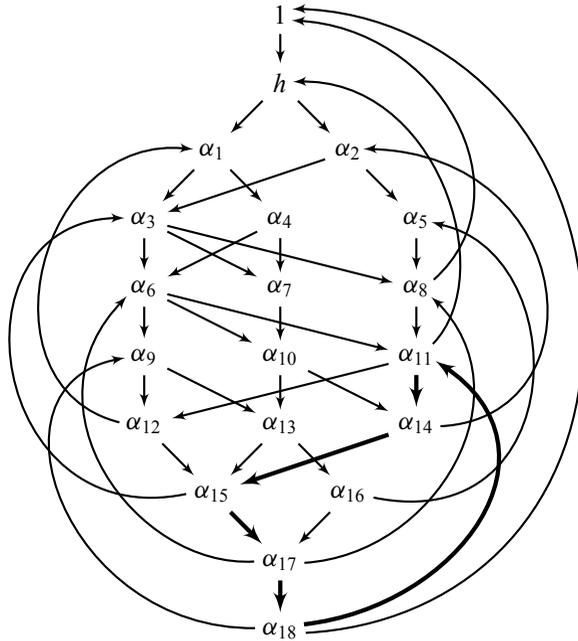
In Proposition 3 below we identify this line with

$$\hat{c}_1(\alpha_1) = 5\alpha_3 + 5\alpha_4 \quad \text{and} \quad \hat{c}_1(\alpha_2) = 10\alpha_3 + 5\alpha_5.$$

**3.1. Case (1) for  $n = 3$ .** We will reformulate the quantum Chevalley formula stated in [Gonzales et al. 2018] using the basis  $\{1, h, \alpha_1, \alpha_2, \dots, \alpha_{18}\}$ .

**Proposition 3.** *The following equalities hold by [Gonzales et al. 2018, Proposition 4.3]:*

- (1)  $\hat{c}_1(1) = 5h$ .
- (2)  $\hat{c}_1(h) = 10\alpha_1 + 5\alpha_2$ .
- (3)  $\hat{c}_1(\alpha_1) = 5\alpha_3 + 5\alpha_4$  and  $\hat{c}_1(\alpha_2) = 10\alpha_3 + 5\alpha_5$ .



**Figure 1.** The quantum Bruhat graph of the Fano variety  $X$  in case (1) for  $n = 3$ .

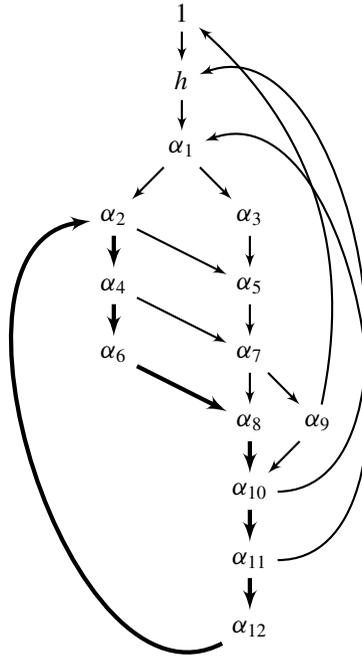
- (4)  $\hat{c}_1(\alpha_3) = 10\alpha_6 + 5\alpha_7 + 5\alpha_8$ ,  $\hat{c}_1(\alpha_4) = 5\alpha_6 + 10\alpha_7$ , and  $\hat{c}_1(\alpha_5) = 5\alpha_8$ .
- (5)  $\hat{c}_1(\alpha_6) = 10\alpha_9 + 5\alpha_{10} + 5\alpha_{11}$ ,  $\hat{c}_1(\alpha_7) = 5\alpha_{10}$ , and  $\hat{c}_1(\alpha_8) = 5\alpha_{11} + 5 \cdot 1$ .
- (6)  $\hat{c}_1(\alpha_9) = 5\alpha_{12} + 5\alpha_{13}$ ,  $\hat{c}_1(\alpha_{10}) = 10\alpha_{13} + 5\alpha_{14}$ ,  $\hat{c}_1(\alpha_{11}) = 5\alpha_{12} + 5\alpha_{14} + 5h$ .
- (7)  $\hat{c}_1(\alpha_{12}) = 5\alpha_{15} + 5\alpha_1$ ,  $\hat{c}_1(\alpha_{13}) = 5\alpha_{15} + 5\alpha_{16}$ , and  $\hat{c}_1(\alpha_{14}) = 5\alpha_{15} + 5\alpha_2$ .
- (8)  $\hat{c}_1(\alpha_{15}) = 5\alpha_{17} + 5\alpha_3$  and  $\hat{c}_1(\alpha_{16}) = 5\alpha_{17} + 5\alpha_5$ .
- (9)  $\hat{c}_1(\alpha_{17}) = 5\alpha_{18} + 5\alpha_6 + 5\alpha_8$ .
- (10)  $\hat{c}_1(\alpha_{18}) = 5\alpha_9 + 5\alpha_{11} + 10 \cdot 1$ .

The quantum Bruhat graph is shown in Figure 1. The bold edges indicate a cycle of length  $r = 5$ , the Fano index.

**Lemma 4.** *Property  $\mathcal{O}$  holds when  $X$  is case (1) with  $n = 3$  of Pasquier’s list.*

*Proof.* The coefficients that appear in the equations in Proposition 3 are the entries of the matrix representation of  $\hat{c}_1$ . Therefore, the matrix representation of  $\hat{c}_1$  is nonnegative. The quantum Bruhat graph is strongly connected by Figure 1, and the cycle  $\alpha_{18}\alpha_{11}\alpha_{14}\alpha_{15}\alpha_{17}\alpha_{18}$  has length  $r = 5$ . □

**3.2. Case (2).** Again, we reformulate the quantum Chevalley formula from [Gonzales et al. 2018] using the basis  $\{1, h, \alpha_1, \alpha_2, \dots, \alpha_{12}\}$ .



**Figure 2.** The quantum Bruhat graph for case (2).

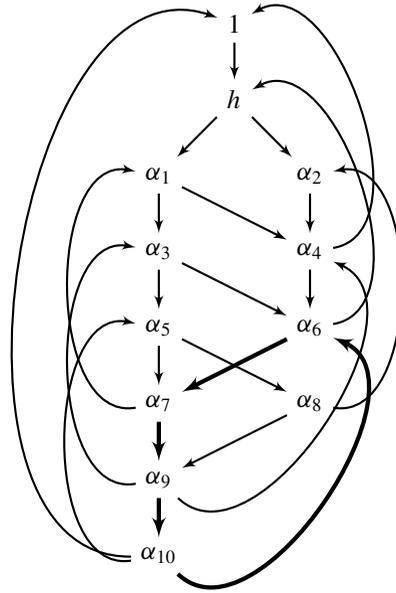
**Proposition 5.** *The following equalities hold by [Gonzales et al. 2018, Proposition 4.4]:*

- (1)  $\hat{c}_1(1) = 7h.$
- (2)  $\hat{c}_1(h) = 7\alpha_1.$
- (3)  $\hat{c}_1(\alpha_1) = 14\alpha_2 + 7\alpha_3.$
- (4)  $\hat{c}_1(\alpha_2) = 7\alpha_4 + 7\alpha_5$  and  $\hat{c}_1(\alpha_3) = 7\alpha_5.$
- (5)  $\hat{c}_1(\alpha_4) = 7\alpha_6 + 7\alpha_7$  and  $\hat{c}_1(\alpha_5) = 7\alpha_7.$
- (6)  $\hat{c}_1(\alpha_6) = 7\alpha_8$  and  $\hat{c}_1(\alpha_7) = 7\alpha_8 + 7\alpha_9.$
- (7)  $\hat{c}_1(\alpha_8) = 7\alpha_{10}$  and  $\hat{c}_1(\alpha_9) = 7\alpha_{10} + 7 \cdot 1.$
- (8)  $\hat{c}_1(\alpha_{10}) = 7\alpha_{11} + 7h.$
- (9)  $\hat{c}_1(\alpha_{11}) = 7\alpha_{12} + 7\alpha_1.$
- (10)  $\hat{c}_1(\alpha_{12}) = 7\alpha_2.$

The quantum Bruhat graph is shown in Figure 2.

**Lemma 6.** *Property  $\mathcal{O}$  holds when  $X$  is case (2) of Pasquier’s list.*

*Proof.* The coefficients that appear in the equations in Proposition 5 are the entries of the matrix representation of  $\hat{c}_1$ . Therefore, the matrix representation of  $\hat{c}_1$  is



**Figure 3.** The quantum Bruhat graph for case (5).

nonnegative. The quantum Bruhat graph is strongly connected by Figure 2, and the cycle  $\alpha_{12}\alpha_2\alpha_4\alpha_6\alpha_8\alpha_{10}\alpha_{11}\alpha_{12}$  has length  $r = 7$ . □

**3.3. Case (5).** Again, we reformulate the quantum Chevalley formula from [Gonzales et al. 2018] using the basis  $\{1, h, \alpha_1, \alpha_2, \dots, \alpha_{10}\}$ .

**Proposition 7.** *The following equalities hold by [Gonzales et al. 2018, Proposition 4.6]:*

- (1)  $\hat{c}_1(1) = 4h$ .
- (2)  $\hat{c}_1(h) = 12\alpha_1 + 4\alpha_2$ .
- (3)  $\hat{c}_1(\alpha_1) = 8\alpha_3 + 4\alpha_4$  and  $\hat{c}_1(\alpha_2) = 4\alpha_4$ .
- (4)  $\hat{c}_1(\alpha_3) = 12\alpha_5 + 4\alpha_6$  and  $\hat{c}_1(\alpha_4) = 4\alpha_6 + 4 \cdot 1$ .
- (5)  $\hat{c}_1(\alpha_5) = 4\alpha_7 + 4\alpha_8$  and  $\hat{c}_1(\alpha_6) = 8\alpha_7 + 4h$ .
- (6)  $\hat{c}_1(\alpha_7) = 4\alpha_9 + 4\alpha_1$  and  $\hat{c}_1(\alpha_8) = 4\alpha_9 + 4\alpha_2$ .
- (7)  $\hat{c}_1(\alpha_9) = 4\alpha_{10} + 4\alpha_3 + 4\alpha_4$ .
- (8)  $\hat{c}_1(\alpha_{10}) = 4\alpha_5 + 4\alpha_6 + 8 \cdot 1$ .

The associated quantum Bruhat graph is shown in Figure 3.

**Lemma 8.** *Property  $\mathcal{O}$  holds when  $X$  is case (5) of Pasquier’s list.*

*Proof.* The coefficients that appear in the equations in Proposition 7 are the entries of the matrix representation of  $\hat{c}_1$ . Therefore, the matrix representation of  $\hat{c}_1$  is nonnegative. The quantum Bruhat graph is strongly connected by Figure 3, and the cycle  $\alpha_{10}\alpha_6\alpha_7\alpha_9\alpha_{10}$  has length  $r = 4$ .  $\square$

Theorem 1 follows from Lemmas 4, 6, 8, and the previously mentioned work done by Li, Mihalcea, Shifler [Li et al. 2019] for the odd symplectic Grassmannian case.

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### References

- [Cheong and Li 2017] D. Cheong and C. Li, “On the conjecture  $\mathcal{O}$  of GGI for  $G/P$ ”, *Adv. Math.* **306** (2017), 704–721. MR Zbl
- [Galkin et al. 2016] S. Galkin, V. Golyshev, and H. Iritani, “Gamma classes and quantum cohomology of Fano manifolds: gamma conjectures”, *Duke Math. J.* **165**:11 (2016), 2005–2077. MR Zbl
- [Gonzales et al. 2018] R. Gonzales, C. Pech, N. Perrin, and A. Samokhin, “Geometry of horospherical varieties of Picard rank one”, preprint, 2018. arXiv
- [Hu et al. 2019] J. Hu, H.-Z. Ke, C. Li, and T. Yang, “Gamma conjecture I for del Pezzo surfaces”, preprint, 2019. arXiv
- [Ke 2018] H.-Z. Ke, “On Conjecture  $\mathcal{O}$  for projective complete intersections”, preprint, 2018. arXiv
- [Li et al. 2019] C. Li, L. C. Mihalcea, and R. M. Shifler, “Conjecture  $\mathcal{O}$  holds for the odd symplectic Grassmannian”, *Bull. Lond. Math. Soc.* **51**:4 (2019), 705–714. MR Zbl
- [Minc 1988] H. Minc, *Nonnegative matrices*, John Wiley & Sons, New York, 1988. MR Zbl
- [Pasquier 2006] B. Pasquier, *Variétés horosphériques de Fano*, Ph.D. thesis, Université Joseph Fourier, 2006, available at <https://tel.archives-ouvertes.fr/tel-00111912/document>.
- [Pasquier 2009] B. Pasquier, “On some smooth projective two-orbit varieties with Picard number 1”, *Math. Ann.* **344**:4 (2009), 963–987. MR Zbl

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