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Given a graph Γ, one can associate a right-angled Coxeter group W and a cube complex Σ on which W acts. By identifying W with the vertex set of Σ, one obtains a growth series for W defined as $W(t) = \sum_{w \in W} t^{\ell(w)}$, where $\ell(w)$ denotes the minimum length of an edge path in Σ from the vertex 1 to the vertex w. The series $W(t)$ is known to be a rational function. We compute some examples and investigate the poles and zeros of this function.

1. Introduction

1.1. ***Preliminaries.*** Throughout this paper, graphs will be undirected, simple, and without loops. It is customary to think of a graph Γ as a collection of vertices with edges joining certain pairs of vertices; throughout the paper, an edge will be denoted by the *unordered* pair $\{u, v\}$ where u and v are the vertices that it joins. Given a graph Γ, we let $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and edge set, respectively.

A *simplicial complex* with vertex set V is a collection K of subsets of V such that if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$. An element $\sigma \in K$ is called a *simplex*, and another element $\tau \in K$ is a *face of* σ if $\tau \subseteq \sigma$. Typically, the vertex set V is a collection of points in $\mathbb{R}^n$, in which case we can identify a simplex σ with its convex hull, obtaining a *geometric realization of* K as a subspace of $\mathbb{R}^n$. Because of this geometric connection, we define the *dimension of a simplex* σ to be $|\sigma| - 1$, where $|\sigma|$ denotes the cardinality of σ. The *dimension* of a simplicial complex is the dimension of a largest simplex. A simplicial complex K is a *flag complex* if whenever all of the edges of a simplex σ are in K, then so is σ. Any graph Γ is a 1-dimensional simplicial complex (on its vertex set $V = V(\Gamma)$), but is a flag complex only if it has no complete subgraphs. Given any graph Γ, there is a unique flag simplicial complex K_Γ that has Γ as its 1-skeleton. It is obtained from Γ by adding subsets $\sigma \subseteq V(\Gamma)$ whenever the induced subgraph on σ is a complete

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subgraph (i.e., a *clique*). We shall call the complex K_Γ the *flag completion* of the graph Γ.

Another standard term we shall assume is that of the *link of simplex σ in K*. By definition, this is the subcomplex $\mathrm{Lk}(\sigma) \subseteq K$ defined by

$$\mathrm{Lk}(\sigma) = \{\tau \in K \mid \sigma \cup \tau \in K \text{ and } \sigma \cap \tau = \varnothing\}.$$

The link is useful because it records the local geometry of K in a neighborhood of σ.

1.2. ***Right-angled Coxeter groups and the Davis complex.*** Given a graph Γ, there is a corresponding *right-angled Coxeter group* W_Γ defined by a presentation with generating set $V(\Gamma)$ and relations

(1) $s^2 = 1$ for every $s \in V(\Gamma)$, and

(2) $st = ts$ for every edge $\{s, t\} \in E(\Gamma)$.

For any such group W_Γ, there is also a CAT(0)-cube complex Σ_Γ obtained from the Cayley graph for this presentation by "filling in cubes" (that is, when all of the edges of an n-dimensional cube are included, the interior of that cube is also included in the complex). The group W_Γ acts cellularly on Σ_Γ, the action is simply transitive on the vertices of Σ_Γ, and the link of every vertex of Σ_Γ is precisely the flag completion of the graph Γ. The cube complex Σ_Γ is called the *Davis complex* associated to the Coxeter group W_Γ. Further details and properties of W_Γ and Σ_Γ can be found in [Davis 2015].

1.3. ***Growth series.*** Given a graph Γ, let $K = K_\Gamma$ be its flag completion, let $W = W_\Gamma$ be the corresponding right-angled Coxeter group, and let $\Sigma = \Sigma_\Gamma$ be the corresponding Davis complex. Fix a vertex x_0 in Σ, and for any $w \in W$, let $\ell(w)$ denote the minimum number of edges in an edge-path connecting x_0 to $w \cdot x_0$ in Σ. We call $\ell(w)$ the *length* of w. Because the 1-skeleton of Σ coincides with the Cayley graph of W with respect to the generators $V(\Gamma)$, this length $\ell(w)$ is also the *word length of w* with respect to this generating set. We then define the *growth series for Γ (or for W)* to be the power series $W(t)$ defined by

$$W(t) = \sum_{w \in W} t^{\ell(w)}.$$

Given a simplicial complex K, we define the *f-polynomial of K* by

$$f(x) = \sum_{\sigma \in K} x^{|\sigma|}.$$

Note that the degree of the f-polynomial is one more than the dimension of K, and the constant term of $f(x)$ is 1 (corresponding to the empty simplex). Also note that $f(x)$ is a generating function in the sense that the coefficients of $f(x)$ record the number of simplices in K of each dimension. A related polynomial that captures the same data is the *h-polynomial*, defined by

$$h(t) = (1-t)^m f\left(\frac{t}{1-t}\right),$$

where m is one more than the dimension of K. The following proposition relates these polynomials to the growth series.

Proposition 1. *Let Γ be a graph and let $f(t)$ and $h(t)$ be the f-polynomial and h-polynomial of its flag completion K_Γ. Then*

$$W(t) = \frac{1}{f(-t/(1+t))},$$

and (using the identity above)

$$W(t) = \frac{(1+t)^m}{h(-t)}.$$

This formula is a special case of the known formula for the growth series of an arbitrary Coxeter group (not just right-angled). A reference for the latter is [Steinberg 1968, Theorem 1.25 and Corollary 1.29]. In [Scott 2007], the second author showed that this same formula holds not just for the Davis complex of a right-angled Coxeter group, but more generally, it holds for the growth series of *any* CAT(0) cube complex whose vertex links all have the same f-polynomial.

Remark 2. Note that Proposition 1 implies that the growth series of a graph is always (the Maclaurin series for) a rational function. Thus from the formula for $W(t)$ in terms of the h-polynomial, it follows that the radius of convergence of $W(t)$ is the minimal norm of a root of the h-polynomial.

Remark 3. Other investigation of the poles and zeroes of the h-polynomial have also appeared in the literature. A generalization of the Charney–Davis conjecture, predicting the sign of the Euler characteristic for flag triangulations of generalized homology spheres, is the real root conjecture (see [Gal 2005] for a clear statement, background, and counterexamples), which states, roughly, that the zeroes of the h-polynomial of a flag triangulation of a generalized homology sphere are all real.

1.4. ***The join of two simplicial complexes.*** Given simplicial complexes K_1 and K_2 with respective vertex sets V_1 and V_2, the *join of K_1 and K_2*, denoted by $K_1 * K_2$, is the simplicial complex with vertex set $V_1 \cup V_2$ and simplices of the form $\sigma_1 \cup \sigma_2$ for all $\sigma_1 \in K_1$ and $\sigma_2 \in K_2$.

It follows easily from the definition of the f-polynomial that if f_1 and f_2 denote the f-polynomials of K_1 and K_2, then the f-polynomial of the join $K_1 * K_2$ is given by

$$f(x) = f_1(x) f_2(x).$$

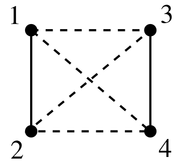

Figure 1. The join of two 1-simplices.

For the example in Figure 1, the complex K is the join of two 1-simplices, which is a 3-simplex. Since the f-polynomial of a 1-simplex is $(x+1)^2$, our formula says that the f-polynomial of K is

$$f(x) = (x+1)^2(x+1)^2 = (x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1.$$

2. Operations on graphs

Several standard graph theory arguments involve the contraction and deletion of edges (see, e.g., chromatic recursion in [West 1996, Theorem 5.3.6]). For flag simplicial complexes, these operations are a little more complicated.

2.1. ***Edge deletion.*** As the local geometry of an edge is fundamental to the way the edge's deletion affects the simplicial complex, it behooves us to encode that geometry in representing how the f-polynomial of the complex changes. We define the subset $L_e \subset K$ of simplices in K which have e as a face, including e itself. Then we define the *local f-polynomial* of an edge e, f_e, as

$$f_e(x) := \sum_{\sigma \in L_e} x^{\dim(\sigma)+1}$$

and the f-polynomial of K after the deletion of e, f_{K-e}, is just $f_K - f_e$. Note that the local f-polynomial of an edge can be calculated by multiplying the f-polynomial of the link of the edge by x^2, as each k-simplex in the link of an edge corresponds to a $(k+2)$-simplex containing that edge in the complex.

The example in Figure 2 shows what happens when K is the simplicial complex on the left and e is the top edge of the shaded 2-simplex. In this case,

$$f_K = x^3 + 5x^2 + 5x + 1,$$

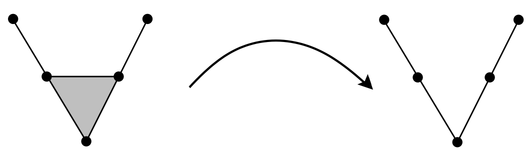

Figure 2. Edge-deletion.

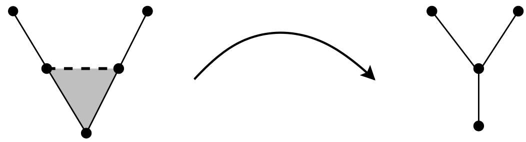

Figure 3. Edge-contraction.

and the link of e is the single vertex at the bottom of the 2-simplex (which has f-polynomial $(x+1)$), so the polynomial f_e is $x^2(x+1) = x^3 + x^2$. It follows that

$$f_{K-e} = f_K - f_e = 4x^2 + 5x + 1,$$

which is the f-polynomial of the (edge-deleted) complex on the right in Figure 2.

2.2. ***Edge contraction.*** The local f-polynomial of an edge can also be used to determine how the contraction of that edge affects the f-polynomial of the simplicial complex. To understand this it is useful to consider how contraction affects simplices with e as a face:

- contracting a 1-simplex removes it from the simplicial complex and identifies its endpoints, thereby removing one 0-simplex;
- contracting an edge of a 2-simplex removes the 2-simplex and identifies its other two edges, thereby removing one 1-simplex;
- contracting an edge of a 3-simplex removes the 3-simplex and identifies the two faces not adjacent to that edge, thereby removing one 2-simplex;

and so on. Thus, for each n-simplex in L_e, contracting e removes one n-simplex and one $(n-1)$-simplex, and therefore the f-polynomial of K after contracting e, $f_{K/e}(x)$, is

$$f_{K/e}(x) = f_K(x) - \left(1 + \frac{1}{x}\right) f_e(x) = f_K(x) - (x + x^2) f_{\mathrm{Lk}(e)}(x).$$

The example in Figure 3 shows what happens when the same edge from the previous example is contracted (instead of deleted). In this case, the f-polynomial for K is

$$f_K = x^3 + 5x^2 + 5x + 1,$$

and the f-polynomial for $\mathrm{Lk}(e)$ is $f_{\mathrm{Lk}(e)} = x + 1$, so our formula for the f-polynomial of K/e is

$$f_{K/e} = (x^3+5x^2+5x+1)-(x^2+x)(x+1) = 3x^2+4x+1.$$

2.3. ***Subdivision.*** The subdivision of a simplicial complex K on an edge e is the complex $\mathrm{Sd}_e(K)$ created by adding a new vertex at the midpoint of e and bisecting all simplices with e as a face. Subdividing a simplicial complex on an edge also

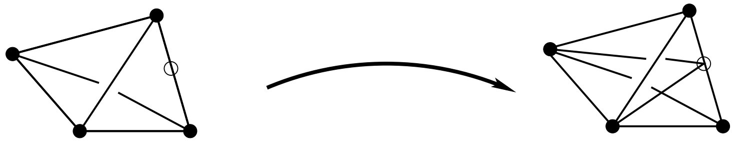

Figure 4. Edge-subdivision of a 3 simplex.

affects the f-polynomial in a way specific to the local geometry of the edge. After subdivision, each n-simplex $\sigma \subset K$ with $e \subseteq \sigma$ becomes two n-simplices bisected by an $(n-1)$-simplex.

Thus the f-polynomial after subdivision is

$$f_{\mathrm{Sd}(e)} = f_K(x) + \left(1 + \frac{1}{x}\right) f_e(x) = f_K(x) + (x + x^2) f_{\mathrm{Lk}(e)}(x).$$

The example in Figure 4 shows what happens when a 3-simplex is subdivided along one of its edges. In this case, K is the 3-simplex, whose f-polynomial is $f_K = (x+1)^4$, and e is the bisected edge, The link of e is the opposite edge, so $f_{\mathrm{Lk}(e)} = (x+1)$. Our formula gives

$$f_{\mathrm{Sd}(e)} = f_K + x(x+1) f_{\mathrm{Lk}(e)}(x) = (x+1)^4 + x(x+1)^3 = 2x^4 + 7x^3 + 9x^2 + 5x + 1.$$

Note the inverse relationship between edge subdivision and edge contraction: subdividing a complex on an edge and then contracting one of the halves of the subdivided edge returns the complex to its original state.

2.4. ***Cartesian graph product.*** Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the Cartesian graph product of G_1 and G_2 is the graph $G = (V, E)$, where $V = V_1 \times V_2$ and

$$E = \big\{\{(u_1, u_2), (v_1, v_2)\} \;\big|\; (u_1 = v_1 \wedge \{u_2, v_2\} \in E_2) \vee (\{u_1, v_1\} \in E_1) \wedge (u_2 = v_2)\big\},$$

with $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$. Let the f-polynomial of graph A be

$$f_A(x) = 1 + \sum_{i \in \mathbb{Z}_{\geq 0}} a_i x^{i+1}, \quad a_i \in \mathbb{Z}_{\geq 0},$$

and the f-polynomial of graph B be

$$f_B(x) = 1 + \sum_{i \in \mathbb{Z}_{\geq 0}} b_i x^{i+1}, \quad b_i \in \mathbb{Z}_{\geq 0}.$$

The Cartesian graph product $A \times B$ then has f-polynomial

$$f_{AB}(x) = b_0 f_A(x) + a_0 f_B(x) - (a_0 b_0 x + a_0 + b_0 - 1).$$

Some intuition for this formula comes from the fact that the product $A \times B$ contains a copy of A for each vertex of B and a copy of B for each vertex of A. The subtracted term corresponds to double-counted edges.

3. Examples and computations

3.1. *Trees.* A connected graph with n vertices and $n-1$ edges is a tree. The f-polynomial of such a tree is

$$f(x) = 1 + nx + (n-1)x^2 = (1 + (n-1)x)(1+x).$$

The h-polynomial is then

$$h(t) = 1 + (n-2)t,$$

the root of which is $1/(2-n)$. It follows (from Remark 2) that for any infinite family of trees, the radius of convergence of the growth series can be arbitrarily close to zero.

3.2. *Cycles.* Cycles are a basic family of graphs and a straightforward example for f- and h-polynomial computation. The k-cycle has k vertices and k edges; thus the f-polynomial for the k-cycle with $k > 3$ is $f(x) = 1 + kx + kx^2$, which has roots

$$x = \frac{-k \pm \sqrt{k^2 - 4k}}{2k}.$$

As $k \to \infty$, these roots approach 0 and -1. We can find the h-polynomial by substituting $x = t/(1-t)$ as follows:

$$\begin{aligned} f\left(\frac{t}{1-t}\right) &= 1 + \frac{kt}{1-t} + \frac{kt^2}{(1-t)^2} = \left(\frac{1}{(1-t)^2}\right)((1-t)^2 + kt(1-t) + kt^2) \\ &= \left(\frac{1}{(1-t)^2}\right)(1 - 2t + t^2 + kt - kt^2 + kt^2), \end{aligned}$$

so

$$h(t) = 1 + (k-2)t + t^2.$$

The roots now are

$$t = \frac{2 - k \pm \sqrt{(k-2)^2 - 4}}{2} = \frac{2 - k \pm \sqrt{k^2 - 4k}}{2},$$

which approach 1 and $-\infty$.

3.3. *Paws.* Define the *paw* as the simplicial complex constructed by attaching an edge to a vertex of a 2-simplex. More generally, a *k-paw* is a 2-simplex with a path of length k (i.e., a path with k edges) attached to a vertex of this 2-simplex (see Figure 5).

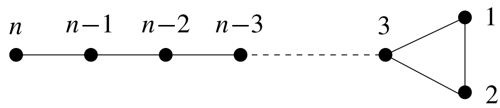

Figure 5. A k-paw with $k = n - 3$.

The f-polynomial for the k-paw is $f(x) = 1 + (k+3)x + (k+3)x^2 + x^3$. We can then find the roots: $f(x)$ factors neatly into $(1+x)(1+(k+2)x+x^2)$, and the quadratic roots are

$$x = \frac{-k-2 \pm \sqrt{(k+2)^2-4}}{2} = \frac{-k-2 \pm \sqrt{k^2+4k}}{2}$$

with limits $\lim_{k\to\infty} x = -1, -\infty$

Likewise, we can derive the h-polynomial using substitution:

$$\begin{aligned} f\left(\frac{t}{1-t}\right) &= 1 + (k+3)\left(\frac{t}{1-t}\right) + (k+3)\left(\frac{t}{1-t}\right)^2 + \left(\frac{t}{1-t}\right)^3 \\ &= \left(\frac{1}{1-t}\right)^3 \big((1-t)^3 + (k+3)t(1-t)^2 + (k+3)t^2(1-t) + t^3\big), \end{aligned}$$

so

$$\begin{aligned} h(t) &= 1-3t+3t^2-t^3+(k+3)t-2(k+3)t^2+(k+3)t^3+(k+3)t^2-(k+3)t^3+t^3 \\ &= 1+kt-kt^2, \end{aligned}$$

with roots

$$t = 1, \ \frac{2-k \pm \sqrt{k^2-4k}}{2}$$

and limits $\lim_{k\to\infty} x = 1, -\infty$.

3.4. ***Cycles with extra diagonals.*** Other families of graphs can be obtained by adding additional diagonals to a cycle. For example, Figure 6 shows a 12-cycle with additional edges connecting vertex i to vertex j whenever $i - j \equiv 2$ or $3 \bmod 12$. This complex has f-polynomial $f(x) = 1 + 12x + 36x^2 + 36x^3 + 12x^4$, with roots $x \approx -0.414, -0.124, -1.231 \pm 0.330i$. Substitution produces the h-polynomial $h(t) = 1 + 8t + 6t^2 - 4t^3 + t^4$, with roots $t \approx -0.706, -0.141, 2.424 \pm 2.033i$. Notably, both the f-polynomial and the h-polynomial for this graph have complex roots.

3.5. ***Possible roots of f- and h-polynomials.*** As f-polynomials have restrictions on their coefficients, it is possible to find restrictions on their possible roots. The first of these is provided by Descartes' rule of signs, which says that the maximum number of positive real roots of a polynomial is less than or equal to the number of sign changes between consecutive coefficients. Since f-polynomial coefficients

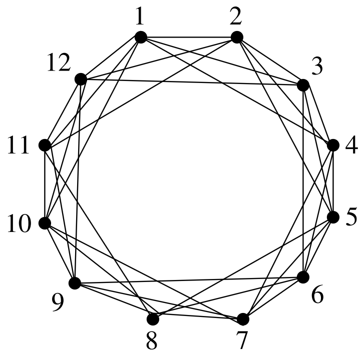

Figure 6. 12-gon with extra diagonals.

count simplices in a complex, they are always nonnegative; thus all roots of f-polynomials must be nonpositive. Due to the fact that, by definition, any simplicial complex contains exactly one -1-simplex, f-polynomials always have 1 as a constant term, and therefore 0 is not a possible root of an f-polynomial. Thus all roots of f-polynomials are negative.

Proposition 4. *The only possible integer root of the h-polynomial of a simplicial complex is* 1.

Proof. Assume the h-polynomial of a simplicial complex K has an integer root $a \in \mathbb{Z}$. Then $x = a/(1-a)$ is a root of the f-polynomial. Note that if a is an integer $\neq 1$, then $x = a/(1-a)$ is a reduced fraction. Since the f-polynomial has integer coefficients, the rational root theorem states that for any rational root of the f-polynomial, written in lowest terms p/q, p is an integer divisor of the constant term of the f-polynomial. Since $p = a$, we know a is an integer divisor of 1, so a must be 1. □

For most of the examples above, the radius of convergence of the growth series $W(t)$ (or, equivalently, the smallest norm of a root of the h-polynomial) is ≤ 1. This actually holds more generally. (The authors thank Michael Hartglass of the Department of Mathematics and Computer Science at Santa Clara University for one of the key ideas in the proof of the following theorem). The key technical tool in the proof is the use of the root test to bound the radius of convergence for a power series.

Theorem 5. *Let K be a flag simplicial complex, let W be the corresponding right-angled Coxeter group, and let Σ be the Davis complex. Assume further that K is not a single simplex. Then the h-polynomial of K has a real root in the interval* $(-1, 0)$.

Proof. We use the second formula from Proposition 1

$$W(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots = \frac{(1+t)^m}{h(-t)},$$

where a_n denotes the number of vertices in Σ that are (edge-path-) distance n from x_0. Since this series is a rational function defined in a neighborhood of 0, we know that the radius of convergence coincides with both the norm of the smallest root of $h(-t)$ and $1/\lambda$ where $\lambda = \limsup_n |a_n|^{1/n}$. Since K is not a single simplex, Σ is unbounded, so the sequence $a_1, a_2, \ldots$ has infinitely many terms ≥ 1. Hence $\lambda \geq 1$. It follows that $h(-t)$ has a root in the interval $(0, 1/\lambda) \subseteq (0, 1)$, and hence $h(t)$ has a root in the interval $(-1, 0)$. □

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